Market and non-market mechanisms for the optimal allocation of scarce resources

Daniele Condorelli [1]

Department of Economics, University of Essex, Wivenhoe Park, Colchester CO43SQ, UK

ARTICLE INFO

ABSTRACT

A number of identical objects is allocated to a set of privately informed agents. Agents have linear utility in money. The designer wants to assign objects to agents that possess specific traits, but the allocation can only be conditioned on the willingness to pay and on observable characteristics. I solve for the optimal mechanism. The choice between market or non-market mechanisms depends on the statistical linkage between characteristics valued by the designer and willingness to pay.

© 2013 Elsevier Inc. All rights reserved.

1. Introduction

A large amount of scarce resources, ranging from broadcasting rights to scarce medical interventions, is periodically allocated by governments or other public institutions. Allocation methods can be generally classified into two categories: (i) *market mechanisms*, if objects are exchanged for money (e.g., auctions or posted prices); (ii) *non-market mechanisms*, if objects are allocated free of charge (e.g., lotteries and priority lists based on observable characteristics).[2]

Market mechanisms are optimal if maximizing welfare of the recipients is the underlying goal of the allocation. Under a set of standard assumptions (i.e., when income effects are absent) Pareto-efficiency alone mandates that the resources be assigned to those who are willing to pay the most for them. Payments must be requested in order to extract information on the willingness to pay. Nevertheless, non-market mechanisms, often coupled with resale-bans, are used in a large number of circumstances, ranging from the allocation of intensive care facilities to the allocation of tickets for concerts.[3]

Two major questions arise. *Under which conditions are non-market mechanisms preferable to market mechanisms? Why are non-market mechanisms so popular?* I develop a theory that addresses both of them.

In my model the objective of the designer can be different from welfare or revenue maximization (e.g., maximizing the number of lives saved in the context of allocating scarce medical resources). As a consequence, non-market mechanisms may

---

[2] In this paper I will also consider hybrid mechanisms, which combine features of both market and non-market mechanisms (see Evans et al., 2009). I will not analyze mechanisms, such as waiting lines, which exploit individuals' willingness to engage in costly effort (see Condorelli, 2012).

[3] See Okun (1975) and Elster (1992) for details on the use of non-market mechanisms in practice.

also be optimal. In particular, market mechanisms are optimal when willingness to pay for the scarce resource is positively associated to those unobservable traits of the agents which are valued by the designer (e.g., effectiveness of the treatment in the context of allocating scarce medical resources). Conversely, the use of non-market mechanisms may be optimal when the statistical dependence between the characteristics valued by the designer and willingness to pay, coupled with binding incentive constraints, prevents the designer from extracting any useful information. Banning resale is necessary in this case. Otherwise, agents with higher willingness to pay would buy the objects from the recipients, upsetting the initial allocation.[4]

The contribution of this paper is twofold, in response to the two questions above. First, and foremost, my analysis provides guidance on selecting among market and non-market mechanisms. For any given policy objective, the normative question of which type of mechanism should be used becomes an empirical one. A take away is that market mechanisms may be optimal even in cases where the allocation objective is not directly related to standard economic welfare. This point has substantial policy implications but, somewhat surprisingly, it has never been formally stated in the economic literature (at least to my knowledge). For example, consider the allocation of scarce medical resources. The medical profession, and policy-makers alike, appear to be strongly against the idea of assigning scarce medical resources to the highest bidder.[5] Instead, need-based criteria, such as saving the highest number of lives, are classic and long-standing rationing principles.[6] I show that opposition to market mechanisms cannot be motivated solely by arguing that the underlying objective of the allocation is different from welfare or revenue maximization. Such considerations also apply to a number of different allocation problems. For instance, my analysis is relevant for the allocation of places in selective state-funded schools.[7]

The use of non-market mechanisms has been previously motivated appealing to moral principles (e.g., Calabresi and Bobbitt, 1978; Walzer, 1983), or to psychological externalities (e.g., the sentiment of repugnance in Roth, 2007). The second contribution of this paper is to offer a new positive theory that may help explaining observed variation in the choice of mechanisms used to allocate scarce resources. Non-market mechanisms prevail if extracting information on willingness to pay cannot help the designer to achieve its goal, either because of incentive problems or because willingness to pay is totally uninformative. However, whether this force has substantial explanatory power (e.g., it explains why radio-spectrum is auctioned to the highest bidder, while places in selective schools are not rationed using prices) is an empirical question that lies outside the scope of this paper.

The formal apparatus is standard. Agents have linear utility in money and their willingness to pay is determined by a set of observable and unobservable characteristics. The designer has an interest in assigning objects to individuals with certain characteristics, some of which may be private information. Because the designer can only condition the final allocation on willingness to pay, relevant incentive constraints are one-dimensional and the mechanism design problem can be solved using standard techniques (following Myerson, 1981).[8] In particular, incentive constraints imply that the designer must select an allocations that offers, to any given agent, the object with higher probability the higher is his willingness to pay. Therefore, even though willingness to pay contains relevant information, non-market mechanisms are optimal when higher willingness to pay is associated with lower expected payoff to the designer.[9] In this case the designer would like to sort types based on willingness to pay, giving priority to lower types, but the best she can do is to condition the allocation on observable characteristics only.[10]

One important feature of my model is that the designer is only capable of extracting information on the willingness to pay. This is the consequence of two assumptions. First, I assume that objects are identical. If objects were heterogeneous, non-market mechanisms could be used to extract information (e.g., see Hylland and Zeckhauser, 1979; McAfee, 1992, and Budish, 2011). Second, I assume that the designer cannot ask the agents to engage in costly activities, such as spending time in line (see Condorelli, 2012). While these assumptions simplify the analysis substantially, my main insights do not hinge on them. First, the conclusion that mechanisms that do not extract any information can be optimal because of binding incentive constraints would still be valid in a setting in which the designer had more instruments to extract information. Second, even if objects were heterogeneous and the designer was able to screen agents by having them to exert a costly

---

[4] Throughout the text I use the term "willingness to pay" even though the term "ability to pay" is often used in the context of specific applications. Willingness to pay seems more accurate, given that the two terms differ only in cases where an individual faces hard budget constraints. Someone could be *willing to pay* but *unable to pay* only if the liquidation value of his wealth (plus the debt he could obtain with no collateral) exceeds the value of the resource to him. Casual empiricism suggests that this is unlikely in most cases of interest.

[5] For example, surveying methods of allocation for scarce medical interventions, Persad et al. (2009) state: "we do not regard ability to pay as a plausible option for the scarce life-saving interventions we discuss". Resistance to the introduction of monetary markets for organs for transplant is also documented in Becker and Elias (2007) and in Roth (2007).

[6] For example, this principle is behind governmental contingency-plans for the allocation of influenza vaccine (see Emanuel and Wertheimer, 2006) and responses to bioterrorism (see Phillips, 2006).

[7] In the UK admission to selective schools is often based on merit only. See for example the *School Admissions Code* from the UK Department of Education.

[8] Two types of a given agent with the same willingness to pay cannot be screened because they have the same preferences over object-money bundles.

[9] For instance, consider the allocation of influenza vaccines. The value of vaccinating an individual depends, in addition to observable characteristics that make him more prone to contract the disease (e.g., age), on the (unobservable) number of people with which he has close contact during his typical workday. However, the willingness to pay for a vaccine, especially in case of a high-risk disease, will be positively correlated to individual wealth and income. In turn, income and wealth are likely to be inversely correlated with the number of people with which an individual might be exposed in his workday. Hence, a non-market mechanism is likely to be optimal in this case.

[10] The fact that objects are randomly assigned does not mean that the designer is *ex-post* indifferent among recipients. This explain why resale must be banned. Allowing resale induces positive correlation between willingness to pay and ownership, something that the designer wants to avoid when non-market mechanisms are optimal.

effort, pure market mechanisms would remain optimal in many cases of interest (e.g., if the objective of the designer was positively correlated with willingness to pay but negatively correlated with willingness to exert the non-monetary cost).

My techniques are standard but nevertheless my work fills a substantial conceptual gap in the literature. Although there are two large bodies of literature that study properties of market and non-market mechanisms independently, few efforts have been devoted to explaining the existence of both types of mechanisms in the same economic environment.[11] Essentially since Gale and Shapley (1962), virtually an entire literature on market-design without money treats the absence of transferable utility as an exogenous restriction.

The most closely related paper is Che et al. (2013). They compare market and non-market allocation methods for the efficient assignment of a unit mass of goods, to a larger set of wealth constrained agents. In Che et al. (2013) the goal of the designer remains that of allocating the objects to the agents with the highest willingness to pay. Therefore, the channel through which non-market mechanisms emerge is different. In Che et al. (2013) lotteries may outperform markets when agents with high willingness to pay are severely budget constrained. However, allowing resale is always welfare-enhancing. In my model, screening based on the willingness to pay may not be the objective of the designer. Indeed, non-market mechanisms are optimal in my model only when the designer is not interested is assigning the object to the agents with the highest willingness to pay. Furthermore, banning resale is always useful.[12]

Other two papers are closely related: Fernandez and Gali (1999) and Esteban and Ray (2006). Fernandez and Gali (1999) compare the performance of markets and tournaments (i.e., non-market mechanisms where agents engage in costly signaling) as allocation mechanisms in an economy with borrowing constraints. In Esteban and Ray (2006) a government seeks to allocate limited resources to productive sectors. However, both sectoral productivity and wealth are privately known. The government, even if it seeks to assign resources to the most productive sectors, may be confounded by the possibility that both high wealth and true economic desirability create loud lobbying.[13]

My paper is also related to other applied mechanism design papers, in particular to works that examine settings with budget constrained agents (e.g., Che and Gale, 1998; Pai and Vohra, 2009, and Dobzinski et al., 2012) and with multidimensional signal spaces and lower dimensional policy spaces (e.g., McAfee et al., 1988; Armstrong, 1996; Rochet and Chone, 1998; Jehiel and Moldovanu, 2001, and Deneckere and Severinov, 2009). The simple structure of my type-space sidesteps most of the implementation difficulties that emerge in this literature.

2. The model

There are $k \geqslant 1$ indivisible and identical objects and n risk-neutral agents. The objects are scarce, $n > k$, and available in fixed quantity. Let $N \equiv \{1, \ldots, n\}$ denote the set of agents. Agents are heterogeneous in terms of observable and unobservable characteristics. For all $i \in N$, let z_i be a real vector of observable characteristics, which are commonly known, and let $\theta_i \in \Theta_i$ be the real vector of unobservable characteristics, which are private information and drawn from a convex set. Write $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_n)$, $\boldsymbol{\Theta} = \times_i \Theta_i$ and let $\boldsymbol{\theta_{-i}}$ and $\boldsymbol{\Theta_{-i}}$ take the usual meaning. Let F_{θ_i} denotes the prior distribution of the unobservable variables, conditional on the observable characteristics of agent i. Assume that unobservable characteristics are always distributed independently across agents.

If agent i obtains one object and pays m_i his utility is $w(z_i, \theta_i) - m_i$, where w is some commonly known function. Write $w_i(\theta_i) \equiv w(z_i, \theta_i)$. Agents have unit demand and obtaining more than one object provides no extra benefit. If i obtains no object and pays m_i, his utility is $-m_i$. The term $w_i(\theta_i)$ is the *willingness to pay* of agent i of type θ_i. Denote w_i the corresponding random variable, F_{w_i} its distribution and f_{w_i} the density. Assume that f_{w_i} has finite expectation, is strictly positive in some interval $W_i = [\underline{w}_i, \overline{w}_i)$ and equal to zero elsewhere.

I model the allocation as a standard mechanism design problem. The timing is as follows: (1) The designer proposes a mechanism. A mechanism is an arbitrary normal-form game. An outcome of the mechanism consists of (i) a random assignment of objects to agents and (ii) a vector of expected payments from agents to the designer. (2) Agents simultaneously and independently decide whether or not to participate in the mechanism. If an agent decides to opt out he makes no payment and obtains no object. (3) Participating agents play the mechanism under incomplete information about each others' types.[14] (4) The outcome of the mechanism is implemented and no further interaction takes place thereafter.

A deterministic assignment of the objects is a vector $(p_1, \ldots, p_n)$ where $p_i \in \{0, 1\}$ for all $i \in N$ and $\sum_i p_i \leqslant k$. The designer is risk neutral and her utility from a given assignment is:

$$\sum_i v(z_i, \theta_i) p_i,$$

---

[11] The analysis of market mechanisms under incomplete information is often labeled "auction theory" (see Krishna, 2002; Milgrom, 2004 or Klemperer, 2004). The major strand of the literature that examines non-market mechanisms is the one studying two-sided matching (see Roth and Sotomayor, 1990).

[12] My main results would not be substantially affected by the presence of budget constrained agents, if the budgets were independent of other unobserved characteristics of the agents. On the one hand, when non-market mechanisms are optimal in the present model, they would be also optimal in the presence of budget constrained agents. On the other hand, when market mechanisms are optimal in the current model, using non-market mechanisms and allowing resale would sometimes be preferable with budget constraints agents (a complete characterization would follow Che et al., 2013).

[13] Recently, Mestieri (2010) studies the optimal design of an educational system in an environment where borrowing constraints and private information make it difficult to separate ability from willingness to pay.

[14] Because participation of all agents is not an issue, I omit to specify that the designer must offer a mechanism which depends on participation decisions.

where v is some commonly known function. For ease of notation, write $v_i(\theta_i) \equiv v(z_i, \theta_i)$ and call $v_i(\theta_i)$ the *value* of an agent i of type θ_i to the designer. I assume that the corresponding random variable, v_i, is well behaved.[15]

This mechanism design objective, which I call *value maximization*, is general and can accommodate different specifications. The designer maximizes utilitarian welfare if $v_i = w_i$ for all $i \in N$. In this case value *is* willingness to pay and the designer maximizes the sum of utilities.[16] My model, however, is interesting only in cases where value cannot be perfectly inferred from willingness to pay and observable characteristics. Suppose, for instance, that agents are heterogeneous in both their value for an object, θ^1, and their opportunity cost of money, θ^2, and the designer seeks to maximize value. In this case $w_i(\theta_i^1, \theta_i^2) \equiv \frac{\theta_i^1}{1+\theta_i^2}$ and $v_i(\theta_i^1, \theta_i^2) \equiv \theta_i^1$. If capital markets are imperfect, and interest rates are heterogeneous, agents with higher value may exhibit lower willingness to pay. More generally, a model where the willingness to pay is a noisy signal of value is very natural in a wide range of practical applications, ranging from the allocation of places in selective schools to the allocation of licenses to operate a regulated business (when the objective of the designer is to maximize the value of the available resources).[17]

My model is standard but two assumptions, which have not been so far discussed, deserve explicit mention. First, the designer has no way, before he proposes a mechanism, of extracting any information other than that already conveyed by the observable characteristics of the agents. This assumption is not crucial. We can always think of the distribution of types as conveying any residual uncertainty. For instance, when considering the allocation of scarce medical resources, the observable characteristics of the agents should already convey any information that it might have been extracted trough diagnostic tests.

Second, agents cannot engage in post-allocation trading of the objects. This assumption is not innocuous, because the set of implementable outcomes shrinks when resale is permitted (see Zheng, 2002) and therefore the designer weakly benefits from banning resale. At one extreme, if resale is allowed and renegotiation among agents is frictionless, the only implementable outcome is one where the objects are assigned to the agents with the highest willingness to pay. However, for the applications at stake, it seems plausible to grant the designer with the power to ban resale. In fact, resale-bans are often observed in practice when non-market mechanisms are employed.

3. Direct mechanism design

Appealing to the revelation principle, the designer can restrict attention to truthful direct revelation mechanisms where everyone obtains a payoff greater than zero. A direct mechanism is a game where each player submits a report about his private information and an outcome is determined as a function of the reports. Hence a *direct mechanism* $\langle \boldsymbol{p}, \boldsymbol{m} \rangle$ is a set of integrable functions $\{p_i : \boldsymbol{\Theta} \to [0,1]; \ m_i : \boldsymbol{\Theta} \to \mathbb{R}\}_{i=1}^n$, where for all $\boldsymbol{\theta} \in \boldsymbol{\Theta}$ the feasibility condition $\sum_i p_i(\boldsymbol{\theta}) \leqslant k$ is met. Because agents have unit demand the designer restricts attention to mechanisms where $p_i(\boldsymbol{\theta}) \leqslant 1$ for each $\boldsymbol{\theta} \in \boldsymbol{\Theta}$.

A direct mechanism is incentive compatible if and only if revealing the true type is optimal when everyone else does so, and every agent always obtains a payoff greater than or equal to zero (i.e., the payoff from the outside option). Formally, a direct mechanism is *incentive compatible* if, and only if, for each $i \in N$ and $\theta_i \in \Theta_i$:

$$\mathrm{E}_{\boldsymbol{\theta}_{-i}}\big[p_i(\theta_i, \boldsymbol{\theta}_{-i}) w_i(\theta_i) - m_i(\theta_i, \boldsymbol{\theta}_{-i})\big] = \max_{\theta \in \Theta_i} \mathrm{E}_{\boldsymbol{\theta}_{-i}}\big[p_i(\theta, \boldsymbol{\theta}_{-i}) w_i(\theta_i) - m_i(\theta, \boldsymbol{\theta}_{-i})\big] \geqslant 0. \tag{1}$$

The utility functions of two types θ_i' and θ_i'' such that $w_i(\theta_i') = w_i(\theta_i'')$ are the same and represent the same preferences over outcomes. As a consequence, the designer cannot obtain a truthful revelation of the information vector if the mechanism assigns outcomes with different utility index to two different types of the same player who both have the same willingness to pay. Otherwise, both types would just report the type of the two that provides the highest utility. Building on this observation, the designer restricts attention to direct mechanisms whereby $p_i(\theta_i', \boldsymbol{\theta}_{-i}) = p_i(\theta_i'', \boldsymbol{\theta}_{-i})$ and $m_i(\theta_i', \boldsymbol{\theta}_{-i}) = m_i(\theta_i'', \boldsymbol{\theta}_{-i})$ for any $i \in N$, $\boldsymbol{\theta}_{-i} \in \boldsymbol{\Theta}_{-i}$ and $\theta', \theta'' \in \Theta_i$ such that $w_i(\theta_i') = w_i(\theta_i'')$.[18] In light of this, a direct mechanism $\langle \boldsymbol{p}, \boldsymbol{m} \rangle$ can be written as a mapping from $\boldsymbol{W} \equiv W_1 \times \cdots \times W_n$ into outcomes. Let $\boldsymbol{w}_{-i}$ indicate an element of $\times_{j \neq i} W_j$.

Define the *interim allocation* as $P_i(w_i) = \mathrm{E}_{\boldsymbol{w}_{-i}}[p_i(w_i, \boldsymbol{w}_{-i})]$. The following result, which is well known and stated without proof, characterizes implementable allocations.[19]

---

[15] There is no need, for our purposes, to model explicitly how F_i, v_i and w_i depend on z_i. As a consequence the optimal mechanism will depend on F_i, v_i and w_i, rather than on the primitives.

[16] It is well known that a first best can be implemented in this case using a Vickrey–Clarke–Groves mechanism (equivalently, a $k+1$-price auction with no reserve price). The optimal mechanism is a market mechanism.

[17] For example the FCC's main spectrum management goal is *not* to assign broadcasting rights to the firms that can profit the most from them, but "to encourage the highest and best use of spectrum" (see the FCC *Report of the Spectrum Efficiency Working Group*, published on November 15 2002). The fact that willingness to pay for spectrum may not be perfectly aligned to its social value is acknowledged, for instance, in McMillan (1995).

[18] Incentive compatibility requires that types with the same willingness to pay obtain the same expected utility. There is no loss of generality in assuming that both the ex-post allocation and payment rules coincide.

[19] See for example Krishna and Maenner (2001).

Lemma 1. *Take any $\boldsymbol{p} = \{p_i : \boldsymbol{W} \to [0,1]\}_{i=1}^n$. There exists $\boldsymbol{m}$ such that $\langle \boldsymbol{p}, \boldsymbol{m} \rangle$ is an incentive compatible direct mechanism if and only if for all $i \in N$:*

$$P_i(w') \geqslant P_i(w'') \quad \text{for all } w', w'' \in W_i \text{ such that } w' \geqslant w''. \tag{2}$$

While Lemma 1 is silent about whether or not the designer is allowed to run a budget deficit, standard results in mechanisms design show that this is not necessary in this environment. Every outcome that is implementable can be implemented without any payment from the designer to agents. Moreover, every implementable outcome can be implemented using a direct mechanism where reporting the true type is a weakly dominant strategy.[20]

4. Market and non-market mechanisms

The distinction between market and non-market mechanisms is made on the basis of equilibrium outcomes. Define the *interim payment* of a given mechanism as $M_i(w_i) = \mathrm{E}_{\boldsymbol{w}_{-i}}[m_i(w_i, \boldsymbol{w}_{-i})]$. For any incentive compatible mechanism, we must have, for all $i \in N$:

$$M(w_i) = M_i(\underline{w}_i) + w_i P_i(w_i) - \underline{w}_i P_i(\underline{w}_i) - \int\limits_{\underline{w}_i}^{w_i} P_i(x)\,dx,$$

where $M_i(\underline{w}_i)$ is an arbitrary constant, less than or equal to $\underline{w}_i P_i(\underline{w}_i)$ in order to ensure a non-negative payoff from participation. Because the designer is not interested in raising money we can always set $M_i(\underline{w}_i) = 0$. I adopt the following convention:

1. A *market mechanism* is one where, for all $i \in N$, $M_i(w_i) > 0$ for all $w_i \in (\underline{w}_i, \overline{w}_i]$ such that $P(w_i) > 0$ (i.e., all those who expect to obtain an object also expect to pay a positive amount of money).
2. A *non-market mechanism* is one where $M_i(\overline{w}_i) = 0$ for all $i \in N$ (i.e., all objects are always assigned free of charge).[21]
3. An *hybrid mechanism* is a mechanism that does not fit into any of the two categories above.

The definition of market mechanisms encompasses, among others, Vickrey–Clarke–Groves mechanisms, revenue maximizing auctions, and also pure posted prices.[22] The set of non-market mechanisms includes, prominently, lotteries and priority lists based on observable characteristics (e.g. point systems). A posted price mechanism where unsold objects are randomized among the remaining agents is an example of an hybrid mechanism.

Setting $M_i(\underline{w}_i) = 0$ implies that an agent of type $\underline{w}_i > 0$ earns a strictly positive payoff if $P_i(\underline{w}_i) > 0$ (i.e., he expects to obtain an object). Therefore, the designer could implement the same allocation by asking an extra lump-sum payment of $P_i(\underline{w}_i)\underline{w}_i$ to every agent. Imposing a participation fee of this sort would also be helpful in discouraging speculators. My definitions do not depend on this normalization. A non-market mechanism could be defined as a mechanism where the allocation does not depend at all on willingness to pay, that is $P_i(w') = P_i(w''_i)$ for all $w', w'' \in W_i$. Similarly, a market mechanism could be defined as one where $P_i(\underline{w}_i) < P_i(w')$ for all $w' \in W_i \setminus \underline{w}_i$.

5. Results

The designer selects a feasible direct mechanism $\langle \boldsymbol{p}, \boldsymbol{m} \rangle$ that maximizes the following objective function subject to incentive compatibility constraints in (1):

$$\mathrm{E}_{\boldsymbol{\theta}} \left[\sum_{i=1}^n p_i \big(w_1(\theta_1), \ldots, w_n(\theta_n) \big) v_i(\theta_i) \right]. \tag{3}$$

Under complete information the designer attains a *first best*. She maximizes the objective in (3) point-wise by assigning the objects to the k agents with the highest values. In general, however, even the presence of a small uncertainty impairs the ability of the designer to achieve a first-best outcome.[23] If there is any uncertainty left, conditional on knowing willingness to pay, about the identity of the k agents with the highest values, then a first best cannot be attained.

---

[20] Both results can be achieved using the Vickrey payment rule: $m_i(\boldsymbol{w}) = p_i(\boldsymbol{w})w_i - \int_{\underline{w}_i}^{w_i} p_i(x_i, w_{-i})\,dx_i$.

[21] $M_i(\overline{w}_i) = 0$ implies that $M_i(w_i) = 0$ for all $w_i \in (\underline{w}_i, \overline{w}_i]$.

[22] In Che et al. (2013) a market mechanism is one where the objects are sold at the *market-clearing price*. Hence, the set of market mechanisms is larger according to my definition.

[23] The result is not surprising. Jehiel and Moldovanu (2001) show that it is generically impossible to implement an ex-post efficient outcome when individual types are multidimensional.

Proposition 1. *Assume that* (i) *there exist* $\theta, \theta' \in \Theta_i$ *such that* $w_i(\theta) = w_i(\theta')$ *and* $v_i(\theta) > v_i(\theta')$; (ii) *there exists* j *with* $\theta'' \in \Theta_j$ *such that* $v_i(\theta) > v_j(\theta'') > v_i(\theta')$; (iii) *there exist a set of type profiles such that, with positive probability, a single object must be allocated to either* i *or* j. *Under the stated assumptions a first-best outcome cannot be achieved.*

Proof. Assumptions (i) and (ii) imply that there is uncertainty, conditional on the willingness to pay, about who, between i and j, should get the object. Condition (iii) ensures that there are profiles of values for which, at the first best, there is one object that must be allocated to either i or j. Because the designer cannot discriminate between types θ and θ' of player i, the fact that a player j exists such that the designer might prefer j over i whenever i is of type θ implies that a first best cannot be achieved. □

In light of the impossibility result above, I now look for second-best mechanisms (i.e., mechanisms that maximize ex-ante the objective of the designer subject to incentive compatibility constraints). A key step consists in rewriting the objective function in a way that makes it manageable using standard techniques.

Lemma 2.

$$E_{\boldsymbol{\theta}}\left[\sum_{i=1}^{n} p_i\big(w_1(\theta_1), \ldots, w_n(\theta_n)\big)v_i(\theta_i)\right] = E_{\boldsymbol{w}}\left[\sum_{i=1}^{n} p_i(\boldsymbol{w})\,E[v_i|w_i]\right].$$

Proof. Follows from the law of iterated expectations. □

I can now solve the design problem, adapting to this environment the *ironing technique* developed in Myerson (1981).

Definition 1. For all $x \in [0, 1]$ let:

$$H_i(x) = \int_0^x E\big[v_i \mid w_i = F_{w_i}^{-1}(y)\big]dy, \qquad G_i(x) = \text{conv}\langle H_i(x)\rangle, \qquad g_i(x) = G_i'(x),$$

where $\text{conv}\langle\cdot\rangle$ denotes the convex hull of the function.[24] The *priority function* of i is:

$$\lambda_i(w_i) = g_i\big(F_{w_i}(w_i)\big).$$

The intuition behind this construction is simple. The value of assigning an object to an agent with willingness to pay w_i is $E[v_i|w_i]$. When the conditional expected value is increasing in w_i the designer will always want to allocate an object to a type w_i'', whenever she allocates it to a type w_i' with $w_i' < w_i''$. Therefore, incentive constraints are not binding and priority in allocation can coincide with $E[v_i|w_i]$. However, when $E[v_i|w_i]$ is decreasing in a certain interval, the designer must take into account that, if she assigns a certain priority to a type w_i', she cannot assign lower priority to types having higher willingness to pay. Hence, she will optimally pool together types in the interval, assigning to all of them the same average priority.

Theorem 1. *Any incentive compatible mechanism that maximizes* (3) *assigns the objects to the k agents with the highest priority levels. Ties among agents can be broken arbitrarily* (e.g., *via an equal chance lottery, which also ensures equal treatment of equals*).

Proof. The designer's problem has been reduced to the following:

$$\max_{\{p_i:\,[\underline{w}_i,\overline{w}_i]\to[0,1]\}_{i=1}^{n}} E_{\boldsymbol{w}}\left[\sum_{i=1}^{n} p_i(\boldsymbol{w})\,E[v_i|w_i]\right]$$

subject to:

$$\sum_{i=1}^{n} p_i(\boldsymbol{w}) \leqslant k \quad \forall \boldsymbol{w} \in \mathbb{R}^n;$$

$$P_i(w) \geqslant P_i(w') \quad \forall i \in N, \ \forall w, w' \in W_i:\ w \geqslant w'.$$

---

[24] G_i is the highest convex function in [0,1] such that $G_i(x) \leqslant H_i(x)$ $\forall x$. Where the derivative of G_i is not defined, we extend it using the right or left derivative.

The candidate solution satisfies the first constraint above. To prove that the second is also satisfied, note that λ_i is the derivative of a convex function and therefore it is monotonically increasing. Then, for all w_{-i}, $p(w)$ is increasing in w_i, which implies that P_i is also increasing.

Next, sum and subtract $P_i(w_i)\lambda_i(w_i)$ inside the objective function and rewrite:

$$\sum_{i=1}^{n} E_{w_i}\big[P_i(w_i)\lambda_i(w_i) + P_i(w_i)\big(E[v_i|w_i] - \lambda_i(w_i)\big)\big].$$

Consider the second term of this expression for every i:

$$\int_{\underline{w}_i}^{\overline{w}_i} P_i(w_i)\big[E[v_i|w_i] - g_i\big(F_{w_i}(w_i)\big)\big]f_{w_i}(w_i)\,dv_i.$$

Integrating it by parts:

$$P_i(w_i)\big[H_i\big(F_{w_i}(w_i)\big) - G_i\big(F_{w_i}(w_i)\big)\big]\big|_{\underline{w}_i}^{\overline{w}_i} - \int_{\underline{w}_i}^{\overline{w}_i}\big[H_i\big(F_{w_i}(w_i)\big) - G_i\big(F_{w_i}(w_i)\big)\big]\,dP_i(w_i).$$

Consider the first term of the above. It is equal to zero: $H_i(0) = G_i(0)$ and $H_i(1) = G_i(1)$, because G_i is the convex hull of the continuous function H_i and thus they coincide at endpoints (the continuity of H_i follows from assuming an atomless F_{w_i}).

With this in mind the objective function becomes:

$$\sum_{i=1}^{n} E_{w}\big[p_i(w)\lambda_i(w_i)\big] - \sum_{i=1}^{n}\int_{\underline{w}_i}^{\overline{w}_i}\big[H_i\big(F_{w_i}(w_i)\big) - G_i\big(F_{w_i}(w_i)\big)\big]\,dP_i(w_i).$$

The candidate solution maximizes the first sum as it puts all the probability on the players for whom $\lambda_i(w_i)$ is maximal. To conclude the proof, we can show that the second term is equal to zero. It must always be non-negative, as $\forall w_i$ $H_i \geqslant G_i$. That it is equal to zero, follows because G_i is the convex hull of H_i and so, whenever $H_i(F_{w_i}(w_i)) > G_i(F_{w_i}(w_i))$, then G_i must be linear and hence $P_i(w_i)$ must be constant. That is, if $G_i(x) < H_i(x)$, $G_i''(x) = g_i'(x) = 0$ and $\lambda_i(w_i)$ will be a constant in a neighborhood of w_i. □

The optimal mechanism takes the form of a ranking among agents based on priority functions that are weakly increasing in willingness to pay. Any incentive compatible mechanism must satisfy this requirement. The theorem adds to the above observation by showing how priority functions are constructed in this context.

Corollary 1. *If* $E[v_i|w_i]$ *is weakly increasing in* w_i *then* $\lambda_i(w_i) = E[v_i|w_i]$.

Proof. Observe that H_i is convex and so $G_i(x) = H_i(x)$ for all $x \in [0, 1]$. □

When $E[v_i|w_i]$ is weakly increasing the optimal mechanism gives priority to the agents with the highest *conditional expected values*. In other words, if there is positive *regression dependence* between value and willingness to pay, it is profitable for the designer to condition the allocation on willingness to pay. Because, for each given agent, the mechanism assigns an object with higher probability the higher is the willingness to pay, incentive constraints are not binding.

When $E[v_i|w_i]$ is strictly increasing and no agent is guaranteed an object given his observable characteristics, all types of agents who expect to obtain an object also expect to make a positive payment. Hence, the optimal mechanism is a market mechanism.[25]

Corollary 2. *Assume that* $E[v_i|w_i]$ *is strictly increasing for all* $i \in N$. *Assume that for each* $i \in N$ *there exist a set* $S \subseteq N$ *of* k *individuals with the property that for all* $j \in S$ *there exists some* $w_j' \in W_j$ *such that* $E[v_j|w_j'] > E[v_i|\underline{w}_i]$. *Then, the optimal mechanism is a market mechanism.*

Proof. Let $\widehat{w}_i = \inf\{w \in W_i : P_i(w_i) > 0\}$ for all i. Then note that $P_i(w_i)$ must be strictly increasing in a right-neighborhood of $\widehat{w}_i$, unless the set above is empty. It follows that $M_i(w_i) > 0$ for all $w_i > \widehat{w}$. □

Agents have a common support of willingness to pay if $\underline{w}_i = \underline{w}_j$ and $\overline{w}_i = \overline{w}_j$ for all $i, j \in N$. Agents are ex-ante symmetric if $F_{w_i} = F_{w_j}$ and, in addition, $v_i(\theta_i)$ is distributed as $v_j(\theta_j)$ for all $i, j \in N$.[26] If agents are ex-ante symmetric and $\mathrm{E}[v_i|w_i]$ is weakly increasing, then any standard auction without a reserve price (i.e. an auction that allocates the objects to the agents with the highest willingness to pay) is an optimal mechanism.[27] In this case, achieving an ex-post Pareto efficient allocation and value maximization are not in contrast. Therefore, the designer need not prohibit resale of the objects. When agents are ex-ante asymmetric but have a common support of willingness to pay, the object will be assigned through an auction which will favor agents that are *observably strong*, in the sense of being more likely to have higher value conditional on having the same willingness to pay.[28]

The above result has policy implications. No matter how value is actually defined, whenever a positive dependence between value and willingness to pay can be established empirically, using non-market mechanisms is a suboptimal policy choice. This suggests that resistance against the use of market mechanisms in some areas of governmental action (e.g., allocation of scarce medical resource) cannot be justified purely by claiming that the objective of the allocation is different from revenue or welfare maximization. One further implication is that auctions should be less strongly favored in cases where the resource or asset has a common value (to the designer). At one extreme, in case the resource being allocated has a common value, all mechanisms are equivalent and a lottery represents the simplest way to allocate the scarce resource.

Corollary 3. *If* $\mathrm{E}[v_i|w_i]$ *is weakly decreasing then* $\lambda_i(w_i) = \mathrm{E}[v_i]$.

Proof. H_i is concave and so G_i is a straight line from $H_i(0) = 0$ to $H_i(1) = \mathrm{E}[v_i]$. $\quad\square$

If, for some agent, the individual value depends negatively on willingness to pay, then the lower is his willingness to pay, the higher is the priority that the designer *would* want to assign him. However, an allocation of this sort is not implementable due to incentive constraints. The second best in this case is to condition the allocation only on observable characteristics and assign the objects to the agents with the highest *unconditional expected values* $\mathrm{E}[v_i]$. The following corollary requires no proof.

Corollary 4. *If* $\mathrm{E}[v_i|w_i]$ *is weakly decreasing for all* $i \in N$, *then the optimal mechanism is a* non-market *mechanism*.

If agents have different expected values then the optimal mechanism is a *priority list* where agents are ranked only on the basis of their observable characteristics. If agents have all the same expected value, all objects can be assigned through an *equal chance lottery*. Banning resale is necessary to prevent post-allocation exchanges, which would reduce the value of the allocation for the designer.

The next example presents a simple generic case in which $\mathrm{E}[v_i|w_i]$ is monotone decreasing and is helpful in providing more intuition on when value and willingness exhibit a negative statistical dependence.

Example 1. Suppose that agents are heterogeneous both in their value for the objects, v_i, and in their opportunity cost of money, r_i, and the designer seeks to maximize value. Let $w_i(v_i, r_i) \equiv \frac{v_i}{1+r_i}$ and $v_i(v_i, r_i) \equiv v_i$. Let the support of (v_i, r_i) be the triangle X_i defined by $v_i \geqslant 2$, $1 \leqslant \frac{v_i}{1+r_i} \leqslant 2$ and $1 + r_i \geqslant \frac{3}{2} v_i - 2$ (see Fig. 1). Let $f_{v_i, r_i} = 1$ in X_i (and zero elsewhere).
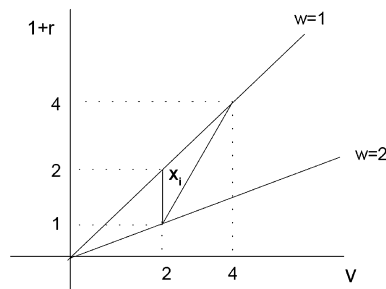

Fig. 1. Type space in Example 1.

After some computations:

$$\mathrm{E}[v_i|w_i] = \frac{16 - 64w_i + 76w_i^2}{12 - 48w_i + 45w_i^2}.$$

This function is strictly decreasing for $1 \leqslant w_i \leqslant 2$. Therefore, if all agents are symmetric, then the optimal mechanism is a lottery.

Corollary 5. *If* $\mathrm{E}[v_i|w_i]$ *is weakly decreasing in some interval* $(w, w') \subset W_i$, *then* λ_i *is constant in* (w, w').

Proof. Note that H_i is concave in $(F_{w_i}(w), F_{w_i}(w'))$ because for a in $(F_{w_i}(w), F_{w_i}(w'))$, $H'(a) = \mathrm{E}[v_i|w_i = F_{w_i}^{-1}(a)]$. Since H_i is concave in $(F_{w_i}(w), F_{w_i}(w'))$, G_i will be a straight line in that interval. It follows that λ_i is constant in (w, w'). $\square$

If s $\mathrm{E}[v_i|w_i]$ is not monotonic for all agents, then the optimal mechanism may be a market mechanism or a hybrid mechanism.

Corollary 6. *Assume that agents are ex-ante symmetric and* $\mathrm{E}[v_i|w_i]$ *is non-monotonic.* (i) *If* $\mathrm{E}[v_i|w_i]$ *is weakly decreasing in a right-neighborhood of* $\underline{w}_i$, *then the optimal mechanism is a* hybrid *mechanism.* (ii) *If* $\mathrm{E}[v_i|w_i]$ *is strictly increasing in a right-neighborhood of* $\underline{w}_i$ *then the optimal mechanism is a* market *mechanism.*

Proof. For (i) observe that because agents are symmetric if the priority function is constant in a right-neighborhood of $\underline{w}_i$, then there is always the possibility that some player obtains an object without making any payment. Hence, for some $w'_i > \underline{w}_i$, we have $P(w'_i) > 0$ and $M(w'_i) = 0$. Because $\mathrm{E}[v_i|w_i]$ is non-monotone there exists an interval of the type space in which λ_i is strictly increasing and therefore there is at least an interval of the type space such that for all w_i in that interval $P(w_i) > 0$ implies $M(w'_i) > 0$. To prove (ii) consider that, if the priority function is increasing in a right-neighborhood of $\underline{w}_i$, then there exists no $w'_i > \underline{w}_i$ such that $P(w'_i) > 0$ and $M(w'_i) = 0$. $\square$

An optimal mechanism in case (i) may be practically implemented as follows. Allocate the objects to the highest bidders and impose a minimum bid. Randomize the remaining goods at no cost among those that did not bid above the minimum bid. This mechanism is optimal if agents are ex-ante symmetric and $\mathrm{E}[v_i|w_i]$ is first decreasing and then increasing. The optimal mechanism in this case has the flavor of ticket pricing for theaters in London or NYC. Tickets are first sold at higher prices to those that do not want to risk missing the show. If the show is not sold-out, remaining tickets are sold on a last minute basis at a substantially discounted price.[29]

An optimal mechanism in case (ii) may be practically implemented as follows. Allocate the objects to the highest bidders but impose a maximum bid. In case of a tie, which will occur with positive probability at the maximum bid, randomize the objects. A mechanism of this type is optimal if agents are ex-ante symmetric and $\mathrm{E}[v_i|w_i]$ is first increasing and then decreasing. This optimal mechanism is practically equivalent to a mechanism where the objects are sold at a price which is below the market price, and where excess demand is rationed using a lottery.[30] This type of mechanisms is used in practice for the allocation of a wide variety of public resources, ranging from housing to hunting and fishing rights (see Scrogin and Berrens, 2003).

In general, an optimal hybrid mechanism under asymmetry may take a number of different forms. One further example is the following. Allocate the objects to all the members of a first group and sell the remaining objects to the highest bidders in a second group. This mechanism might be optimal if the minimum value for an agent in the first group is above the maximum for an agent in the second group and $\mathrm{E}[v_i|w_i]$ is increasing for agents in the second group.

6. Conclusions

In my model, a number of identical objects are allocated to a set of heterogeneous agents. The designer seeks to attribute the objects to agents with specific unobservable characteristics (i.e., seeks to maximize value), but she can only condition the allocation on willingness to pay.

Solving the above mechanism design problem, I provide one main insight. Both market and non-market allocations can be optimal, depending on the regression dependence between the traits that are valued by the designer and willingness to pay. Lotteries, priority lists or other hybrid mechanisms dominate pure market mechanisms if higher willingness to pay signals lower expected value, at least in some range of the type space. In this case incentive constraints prevents the designer

---

[29] See "The Dynamics of Pricing Tickets for Broadway Shows" by Hal Varian (NYTimes 2005-1-13).

[30] The theoretically optimal mechanism would require the designer to run an auction with a maximum bid rather than using posted prices. This is a consequence of the fact that the designer wants to always assign all the objects, even in the unlikely event that there is no excess demand at the maximum bid.

from conditioning the allocation on willingness to pay in a way that is beneficial to her. Conversely, market mechanism are optimal when a positive dependence between value and willingness to pay can be established.

I draw one substantial policy implication from this analysis. Even though the stated criteria for rationing certain goods might be unrelated to willingness to pay of the potential recipients, as far as optimality requires conditioning the allocation on private information, the use of market mechanisms might help achieve better outcomes.

My analysis may also have positive content. Auctions and other market mechanisms are, and have been, widely used for the allocation of rights to exploit scarce natural resources (e.g. spectrum, oil fields, timber, etc.) and for the privatization of state owned assets. My theory is consistent with this observation in the sense that one should expect a strong positive association between the ability to generate social value from a resource and the willingness to pay of potential buyers. Furthermore, it may be argued that a strong case in support of using market mechanisms for allocating scarce medical resources has never been made, because the unobserved individual characteristics that determine the effectiveness of a treatment are unrelated to willingness to pay.[31] However, more generally, whether or not the statistical dependence between value (however defined) and willingness to pay may explain the adoption of market or non-market mechanisms in some domains of policy intervention is an open empirical question that cannot be satisfactorily settled here.

References

Armstrong, Mark, 1996. Multiproduct nonlinear pricing. Econometrica 64 (1), 51–75.
Becker, Gary S., Elias, Julio Jorge, 2007. Introducing incentives in the market for live and cadaveric organ donations. J. Econ. Perspect. 21 (3), 3–24.
Budish, Eric, 2011. The combinatorial assignment problem: Approximate competitive equilibrium from equal incomes. J. Polit. Economy 119 (6), 1061–1103.
Calabresi, Guido, Bobbitt, Philip, 1978. Tragic Choices. W.W. Norton & Company, New York.
Che, Yeon-Koo, Gale, Ian, 1998. Standard auctions with financially constrained bidders. Rev. Econ. Stud. 65 (1), 1–21.
Che, Yeon-Koo, Gale, Ian, Kim, Jinwoo, 2013. Assigning resources to budget-constrained agents. Rev. Econ. Stud. 80 (1), 73–107.
Condorelli, Daniele, 2012. What money can't buy: Efficient mechanism design with costly signals. Games Econ. Behav. 75 (2), 613–624.
Deneckere, Raymond, Severinov, Sergei, 2009. Multi-dimensional screening with a one-dimensional allocation space. Mimeo.
Dobzinski, Shahar, Lavi, Ron, Nisan, Noam, 2012. Multi-unit auctions with budget limits. Games Econ. Behav. 74 (2), 486–503.
Elster, Jon, 1992. Local Justice. Russel Sage Foundation, New York.
Emanuel, Ezekiel, Wertheimer, Alan, 2006. Who should get influenza vaccine when not all can? Science 312, 854–855.
Esteban, Joan, Ray, Debraj, 2006. Inequality, lobbying, and resource allocation. Amer. Econ. Rev. 96 (1), 257–279.
Evans, Mary F., Vossler, Christian A., Flores, Nicholas E., 2009. Hybrid allocation mechanisms for publicly provided goods. J. Public Econ. 93 (1–2), 311–325.
Fernandez, Raquel, Gali, Jordi, 1999. To each according to…? Markets, tournaments, and the matching problem with borrowing constraints. Rev. Econ. Stud. 66 (4), 799–824.
Gale, David, Shapley, Lloyd, 1962. College admissions and the stability of marriage. Amer. Math. Monthly 69 (1), 9–15.
Hoppe, Heidrun C., Jehiel, Philippe, Moldovanu, Benny, 2006. License auctions and market structure. J. Econ. Manag. Strategy 15 (2), 371–396.
Hylland, Aanund, Zeckhauser, Richard, 1979. The efficient allocation of individuals to positions. J. Polit. Economy 87 (2), 293–314.
Jehiel, Philippe, Moldovanu, Benny, 2001. Efficient design with interdependent valuations. Econometrica 69 (5), 1237–1259.
Klemperer, Paul, 2004. Auctions: Theory and Practice. Princeton University Press.
Krishna, Vijay, 2002. Auction Theory. Academic Press.
Krishna, Vijay, Maenner, Eliot, 2001. Convex potentials with an application to mechanism design. Econometrica 69 (4), 1113–1119.
McAfee, Preston, 1992. Amicable divorce: Dissolving a partnership with simple mechanisms. J. Econ. Theory 56, 266–293.
McAfee, Preston, R., McMillan, John, 1988. Multidimensional incentive compatibility and mechanism design. J. Econ. Theory 46 (2), 335–354.
McMillan, John, 1995. Why auction spectrum? Telecommunications Pol. 19 (3), 191–199.
Mestieri, Marti, 2010. Wealth distribution and human capital: How borrowing constraints shape educational systems. Mimeo.
Milgrom, Paul, 2004. Putting Auction Theory to Work. Cambridge University Press.
Myerson, Roger, 1981. Optimal auction design. Math. Operations Res. 6 (1), 58–73.
Okun, Arthur M., 1975. Equality and Efficiency: The Big Tradeoff. Brookings Institution Press, Washington.
Pai, Mallesh, Vohra, Rakesh, 2009. Optimal auctions with financially constrained bidders. Mimeo.
Persad, Goving, Wertheimer, Alan, Emanuel, Ezekiel, 2009. Principles for allocation of scarce medical interventions. Lancet 373, 423–431.
Phillips, S., 2006. Current status of surge research. Acad. Emerg. Med. 13, 1103–1108.
Rochet, Jean-Charles, Chone, Philippe, 1998. Ironing, sweeping, and multidimensional screening. Econometrica 66 (4), 783–826.
Roth, Alvin, 2007. Repugnance as a constraint on markets. J. Econ. Perspect. 21 (3), 37–58.
Roth, Alvin E., Sotomayor, Marilda A., 1990. Two Sided Matching. Cambridge University Press.
Scrogin, David O., Berrens, Robert P., 2003. Rationed access and welfare: The case of public resource lotteries. Land Econ. 79 (2), 137–148.
Walzer, Michael, 1983. Spheres of Justice. Basic Books, New York.
Zheng, Charles Zhoucheng, 2002. Optimal auction with resale. Econometrica 70 (6), 2197–2224.

---

[31] This seems consistent with the view held by the American Medical Association: "Consideration of a patient's ability to pay is problematic in many areas of health care, but especially when it comes to scarce, lifesaving resources. In other areas, a market-based distribution according to ability (or willingness) to pay may accurately reflect individuals' different valuations of various goods and services. At present, though, the disparity among incomes across society distorts the accuracy of the market model as a fair tool for distributing scarce medical resources, for the amount an individual can spend to gain access to a needed treatment will often fall short of his or her actual valuation of it." (see CEJA Report K-A-93, Ethical considerations in the allocation of organs and other scarce medical resources among patients).