DS 574 Algorithmic Mechanism Design	**Lecture #8 Worksheet**
Fall 2022	Prof. Kira Goldner

Walrasian Equilibria in the Unit-Demand Setting

The Unit-Demand Setting: There are m non-identical items U and n bidders where each bidder i has private valuation v_{ij} for each item j. Bidder i is unit demand, that is, wants at most one item for any set S:

$$v_i(S) := \max_{j \in S} v_{ij}.$$

Definition 1. In the unit-demand setting, a *Walrasian equilibrium* (or "competitive equilibrium") is a price vector $\mathbf{q} \in \mathbb{R}^m$ defined on the items and a matching M of the bidders and items such that:

1. Each bidder i is matched to a favorite item $j \in \text{argmax}\{v_{ij} - q_j\}_{j \in U \cup \{\emptyset\}}$. (WE1)

   Equivalently, $\mathbf{q}$ is an *envy-free pricing.*

2. An item $j \in U$ is unsold *only* if $q(j) = 0$. (WE2)

We call $D_i(\mathbf{q}) = \text{argmax}\{v_{ij} - q_j\}_{j \in U \cup \{\emptyset\}}$ the *demand set* of i under prices $\mathbf{q}$.

Claim 1 (First Welfare Theorem)**.** In the unit-demand setting, if $(\mathbf{q}, M)$ is a Walrasian Equilibrium, then M is a welfare-maximizing allocation.

This essentially says "markets are efficient," and there are many "First Welfare Theorems" each with this flavor. Exercise: Prove this.

What we'll now see that is the VCG allocation and payment *is* a WE, and in fact, is a lower bound on all WE for the unit-demand setting.

Recall the VCG payment in this setting:

$$p_i = \sum_{k \neq i} v_k(M^{-i}(k)) - \sum_{k \neq i} v_k(M(k))$$

where $M(k)$ is the item that k is allocated in the welfare-maximizing (maximum-weight) matching, and M^{-i} is the welfare-maximizing matching without bidder i.

Theorem 1 (VCG Payments Lower Bound WE)**.** *In the unit-demand setting, let* $\mathbf{p}$ *denote the induced item price vector of the truthful-revelation VCG outcome and* $\mathbf{q}$ *a Walrasian price vector. Then* $p(j) \leq q(j)$ *for every item* j*.*

Proof. Let M denote the allocation computed by the VCG mechanism. Let M^{-i} denote a welfare-maximizing allocation among allocations that leave bidder i unmatched. The pair $(\mathbf{q}, M)$ is a WE. (Why?) For every $k \neq i$, (WE1) of $(\mathbf{q}, M)$ can be used to argue that k prefers $M(k)$ over $M^{-i}(k)$ at the prices $\mathbf{q}$:

and summing over all $k \neq i$ gives

where $Q = \sum_{j'} q(j')$, because $\sum_{k \neq i} q(M(k))$ sums over all of the items with non-zero q-prices except for the item matches to i (which we call j). Rearranging gives

where the equation follows from the definition of prices from the VCG mechanism. □

Theorem 2 (VCG Outcome is a WE)**.** *In the unit-demand setting, let M and $\mathbf{p}$ denote the allocation and induced item price vector of the truthful-revelation VCG outcome. Then $(\mathbf{p}, M)$ is a WE.*

Then in unit-demand settings, a WE is guaranteed to exists, there is a "smallest" WE, and the VCG outcome is precisely this smallest WE. We leave the proof as an exercise, but you may want to use the following lemma.

Lemma 1. *In the unit-demand setting, let M and $\mathbf{p}$ denote the allocation and induced item price vector of the truthful-revelation VCG outcome. For a good $j \in U$, let M^{+j} denote a welfare-maximizing allocation after adding a second copy j' of the good j (with $v_{ij} = v_{ij'}$ for every bidder i). Then*

$$p(j) = \sum_{k=1}^{n} v_k(M^{+j}) - \sum_{k=1}^{n} v_k(M).$$

The Crawford-Knoer Auction

1. Set a price q_j for each item j, initializing each price to 0.
2. Initially all bidders are unassigned.
3. while (TRUE):
    (a) Ask each bidder for a favorite item (or $\emptyset$) at the prices $\mathbf{q} + \varepsilon$, meaning an item $j \in D_i(\mathbf{q} + \varepsilon) := \text{argmax}_k \{v_{ik} - (q_k + \varepsilon)\}$. Treat this as a "bid" for item j.
    (b) If no unassigned bidder submits a bid, then halt with the current allocation and prices $\mathbf{q}$.
    (c) Otherwise, pick an arbitrary unassigned bidder i that bid for item j and assign j to i.
        i. If item j was previously assigned to bidder i', mark i' as unassigned and increase the price q_j by ε.

Observations:

Analysis of the CK Auction

Theorem 3. *Up to ε terms, the outcome of the CK auction under sincere bidding is the VCG outcome under truthful revelation.*

Lemma 2. *If all bidders bid sincerely, then the CK auction terminates at an ε-WE (q, M).*

Corollary 4. *If all bidders bid sincerely, then the CK auction terminates with an allocation that has surplus within $m\varepsilon$ of the maximum possible.*

Note: Consistent vs. sincere bidding—i's possible actions:

1. Answer all queries honestly (with respect to v_i).
2. For some valuation $v_i' \neq v_i$, answer all queries as *if* its valuation was v_i'.
3. Answer queries in an arbitrary, possibly inconsistent, way. valuation.

(1) and (2) are consistent with respect to *some* valuation.

Proposition 5. *Let $\mathcal{A}$ be an iterative auction such that the sincere bidding outcome of $\mathcal{A}$ is the same as the truthful revelation outcome of the VCG mechanism. For every bidder i and valuation profile $\mathbf{v}$, if every player other than i bids sincerely, then sincere bidding is bidder i's best response* among consistent actions*.*

Theorem 6. *The CK auction is EPIC (up to error $2\varepsilon \cdot \min\{m, n\}$).*

See Tim Roughgarden's notes for an expansion on this section.

The Gross Substitutes Condition

A General Valuation Model

The most general welfare-maximization problem we'll consider in this course is the following.

- There is a set U of m non-identical goods.
- Each bidder $i = 1, 2, 3, \ldots, n$ has a private valuation $v_i(S)$ for each bundle $S \subseteq U$ of goods that it might receive.
  - Assumption #1: $v_i(\emptyset) = 0$.
  - Assumption #2: "free disposal," meaning the monotonicity condition that $v_i(S) \leq v_i(T)$ whenever $S \subseteq T$.

Generalized Walrasian Equilibrium: A Walrasian equilibrium (WE) is a nonnegative price vector $\mathbf{q}$ on the items and an allocation $(S_1, \cdots, S_n)$ such that:

(WE1) Each bidder i is matched to a favorite bundle

$$S \in \text{argmax}\{v_i(S) - \sum_{j \in S} q(j)\}_{S \subseteq U} = D_i(\mathbf{q}),$$

with the empty set $S = \emptyset$ is allowed.

(WE2) An item $j \in U$ is unsold only if $q(j) = 0$.

The Kelso-Crawford Auction

An extension of the CK auction where bidders can bid on more than one item at once, and can also bid for new items even if some items are already assigned to them. It remains impossible to withdraw from a bid.

Kelso-Crawford (KC) Auction:

1. Initialize the price of every item j to $q(j) = 0$.
2. For every bidder i, initialize the set S_i of items assigned to i to $\emptyset$.
3. while (TRUE):
   (a) Ask each bidder for their favorite subset of items not assigned to them, given the items they already have and the current prices—an arbitrary set T_i in

   $$\text{argmax}_{T \subseteq U \setminus S_i} \{v_i(S_i \cup T) - \mathbf{q}^{\varepsilon}(S_i \cup T)\},$$

   where

   $$\mathbf{q}^{\varepsilon}(S_i \cup T) = \sum_{j \in S_i} q(j) + \sum_{j \in T} (q(j) + \varepsilon).$$

   (b) If $T_i = \emptyset$ for all bidders i, then halt with the current allocation $(S_1, \ldots, S_n)$ and prices $\mathbf{q}$.
   (c) Otherwise, pick an arbitrary bidder i with $T_i \neq \emptyset$:
      i. $S_i \leftarrow S_i \cup T_i$;
      ii. for all $k \neq i$, $S_k \leftarrow S_k \setminus T_i$;
      iii. for $j \in T_i$; $q(j) \leftarrow q(j) + \varepsilon$.

In the special case of unit-demand bidders, the KC auction is identical to the CK auction.

For bidders with general valuations, bidding sincerely in the KC auction can be a disaster.

Example:

The Gross Substitutes Condition

Definition 2. A valuation v_i defined on item set U satisfies the *gross substitutes (GS) condition* if and only if the following condition holds. For every price vector $\mathbf{p}$, every set $S \in D_i(\mathbf{p})$, and every price vector $\mathbf{q} \geq \mathbf{p}$, there is a set $T \subseteq U$ with

$$(S \setminus A) \cup T \in D_i(\mathbf{q}),$$

where $A = \{j : q(j) > p(j)\}$ is the set of items whose prices have increased (in $\mathbf{q}$ relative to $\mathbf{p}$).

Theorem 7. *If all bidders have gross substitutes valuations and bid sincerely, then the Kelso-Crawford auction terminates at a $m\varepsilon$-Walrasian equilibrium.*

Proof.

Taking the limit as $\varepsilon \to 0$ gives the following remarkable consequence of the KC auction.

Corollary 8. *If valuations $v_1, \cdots, v_n$ satisfy the gross substitutes condition, then there exists a Walrasian equilibrium.*

Proof Sketch.

Thus far we've been taking for granted the existence of Walrasian equilibrium. In many cases, however, WE do not exist.

Example:

More generally, the gross substitutes condition is in some sense the frontier for the guaranteed existence of WE.

Theorem 9 (Gul-Stacchetti)**.** *If v_i is a valuation that does not satisfy the GS condition, there are unit-demand (and hence GS) valuations $\mathbf{v}_{-i}$ such that $\mathbf{v}$ admits no Walrasian equilibrium.*

An Impossibility Result

Can we get the VCG outcome with GS valuations using an ascending auction? Perhaps surprisingly, the answer is no.

Theorem 10 (Gul-Stacchetti)**.** *There is no ascending auction for which sincere bidding yields the VCG outcome for every profile of gross substitutes valuations.*