DS 574 Algorithmic Mechanism Design	**Lecture #5**
Fall 2022	Prof. Kira Goldner

Single-Parameter Optimal Revenue (continued)

Virtual Welfare Recap

- Maximize welfare ($\sum_i v_i x_i$): Always give the bidder the item, always give it away for free!
- Maximize revenue: Post a price that maximizes $\textsc{Rev} = \max_r r \cdot [1 - F(r)]$.

Using only the revelation principle and the payment identity $p_i(b_i, \mathbf{b}_{-i}) = b_i \cdot x_i(b_i, \mathbf{b}_{-i}) - \int_0^{b_i} x_i(z, \mathbf{b}_{-i})\, dz$, we proved the following:

$$\textsc{Revenue} = \mathbb{E}_{\mathbf{v}\sim\mathbf{F}}[\sum_i p_i(\mathbf{v})] = \mathbb{E}_{\mathbf{v}\sim\mathbf{F}}[\sum_i \varphi_i(v_i) x_i(\mathbf{v})] = \textsc{Virtual Welfare}$$

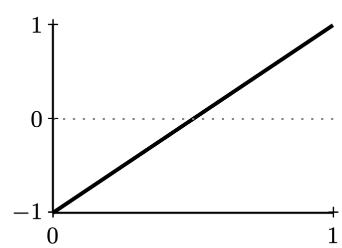
where

$$\varphi_i(v_i) = v_i - \frac{[1 - F_i(v_i)]}{f_i(v_i)}.$$

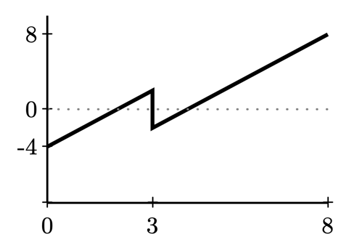
Then similarly to welfare, just give the item to the bidder with the highest (non-negative) *virtual* value! But this doesn't work when $\varphi(\cdot)$ isn't monotone, because then $x(\cdot)$ wouldn't be.

Definition 1. A distribution F is regular if the corresponding virtual valuation function $\varphi(v) = v - \frac{1-F(v)}{f(v)}$ is strictly increasing.

Claim 1. A virtual welfare maximizing allocation x is monotone if and only if the virtual value functions are regular.



(a) Uniform agent virtual value.



(b) Bimodal agent virtual value.

Figure 1: Virtual value functions $\varphi(v) = v - \frac{1-F(v)}{f(v)}$ for the uniform and bimodal agent examples.

It will be helpful to keep the following two examples in mind:

a. a uniform agent with $v \sim U[0, 1]$. Then $F(x) = x$ and $f(x) = 1$. $\varphi(v) = 2v - 1$.

b. a bimodal agent with

$$v \sim \begin{cases} U[0,3] & w.p. \frac{3}{4} \\ U(3,8] & w.p. \frac{1}{4} \end{cases} \qquad \text{and} \qquad f(v) = \begin{cases} \frac{3}{4} & v \in [0,3] \\ \frac{1}{20} & v \in (3,8] \end{cases}$$

$$1 - F(v) = \begin{cases} \frac{1}{4} + \left(\frac{3-v}{3}\right) \cdot \frac{3}{4} & v \in [0,3] \\ \left(\frac{8-v}{5}\right) \cdot \frac{1}{4} & v \in (3,8] \end{cases} \qquad \text{so} \qquad \varphi(v) = \begin{cases} \frac{4}{3}(v-1) & v \in [0,3] \\ 2v - 8 & v \in (3,8] \end{cases}$$

Quantile Space and Ironing

Instead of talking in *value space*, where an agent has value v, the fraction of the distribution with value above v is $1-F(v)$, and the revenue from posting a "take-it-or-leave-it" price of v is $v[1-F(v)]$, we will instead talk about *quantiles*.

Let $1 - F(v) = q$, the fraction of the distribution with a value at least v, willing to pay a price of v. *Quantile* q refers to the the fraction of the distribution left above its corresponding value. For example, consider a distribution that is $U[\$0, \$10]$. Then the quantile 0.1 corresponds to \$9, where 10% of the population might have a higher value. We let $v(q)$ denote the corresponding value, so $v(0.1)$ is \$9.

Definition 2. The *quantile* of a single-dimensional agent with value $v \sim F$ is the measure with respect to F of stronger values, i.e., $q = 1 - F(v)$; the inverse demand curve maps an agent's quantile to her value, i.e., $v(q) = F^{-1}(1-q)$.

Quantiles are particularly useful because we can draw an agent from any distribution by drawing a quantile $q \sim U[0,1]$. That is, for any $\hat{q}$ and any distribution F, $\Pr_F[q \leq \hat{q}] = \hat{q}$. In English: the probability that an agent has a value in the top 0.3 of the distribution is 0.3.

For everything we do today, we *could* stay in value space, but we'd have to normalize by the distribution using $f(v)$, which makes everything a bit messier and a bit trickier.

Example: For the example of a uniform agent where $F(z) = z$, the inverse demand curve is $v(q) = 1 - q$.

For an allocation rule $x(\cdot)$ in value space, we define an allocation rule in *quantile space* $y(\cdot)$:

$$y(q) = x(v(q)).$$

As $x(\cdot)$ is monotone weakly increasing, then $y(\cdot)$ is monotone *weakly decreasing*.

Definition 3. The *price-posting revenue curve* of a single-dimensional linear agent specified by inverse demand curve $v(\cdot)$ is $P(q) = q \cdot v(q)$ for any $q \in [0,1]$.

Assuming the lower-end of the support of F is 0 and the upper end is some finite $v_{\max}$, then $P(0) = 0$ and $P(1) = 0$.

Claim 2. Any allocation rule $y(\cdot)$ can be expressed as a distribution of posted prices.

Proof. Given the allocation rule $y(\cdot)$, consider the distribution $G^y(z) := 1 - y(z)$. We show that the mechanism that randomly draws a quantile $\hat{q} \sim G^y$ from the distribution G^y and posts the price $v(\hat{q})$ is equivalent.

For a random price $v(\hat{q})$ and fixed quantile q, then

$$\Pr_{\hat{q}\sim G^y}[v(\hat{q}) < v(q)] = \Pr_{\hat{q}\sim G^y}[\hat{q} > q] = 1 - G^y(q) = y(q).$$

□

Claim 3. A distribution F is regular if and only if its corresponding price-posting revenue curve is concave.

Observe that $P'(q) = \varphi(v(q))$:

$$P'(q) = \frac{d}{dq}(q \cdot v(q)) = v(q) + qv'(q) = v - \frac{1 - F(v)}{f(v)} = \varphi(v(q)).$$

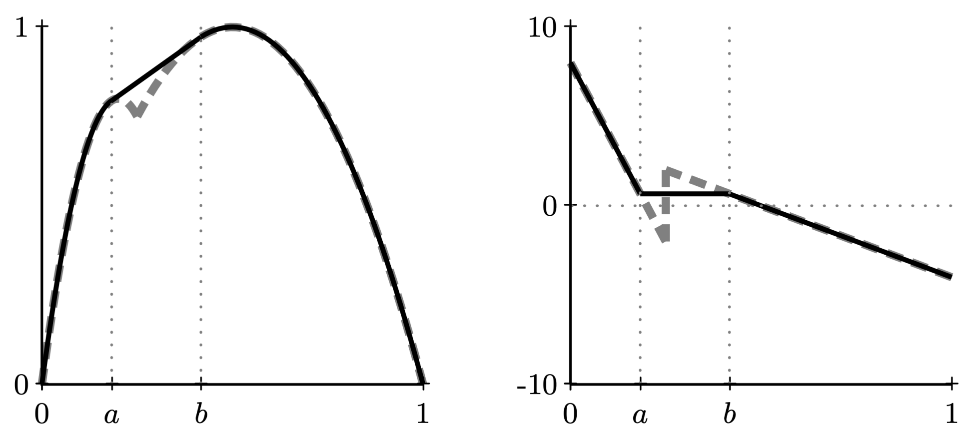
Thus $\Phi(q) = \int_0^q \varphi(\hat{q})\, d\hat{q} = P(q)$.

Definition 4. The *ironing procedure* for (non-monotone) virtual value function φ (in quantile space) is:

(i) Define the cumulative virtual value function as $\Phi(\hat{q}) = \int_0^{\hat{q}} \varphi(q)\, dq$.

(ii) Define ironed cumulative virtual value function as $\bar{\Phi}(\cdot)$ as the concave hull of $\Phi(\cdot)$.

(iii) Define the ironed virtual value function as $\bar{\varphi}(q) = \frac{d}{dq}\bar{\Phi}(q) = \bar{\Phi}'(q)$.



(a) Ironed cumulative virtual values. (b) Ironed virtual values.

Figure 2: The bimodal agent's (ironed) revenue curve and virtual values in quantile space.

Theorem 1. *For any monotone allocation rule $y(\cdot)$ and any virtual value function $\varphi(\cdot)$, the expected virtual surplus of an agent is upper-bounded by her expected ironed virtual surplus, i.e.,*

$$\mathbb{E}[\varphi(q)y(q)] \leq \mathbb{E}[\bar{\varphi}(q)y(q)].$$

Furthermore, this inequality holds with equality if the allocation rule y satisfies $y'(q) = 0$ for all q where $\bar{\Phi}(q) > \Phi(q)$.

Proof. Recall integration by parts:

$$\int_a^b u(x)v'(x)\,dx = [u(x)v(x)]_a^b - \int_a^b u'(x)v(x)\,dx.$$

By integration by parts for any virtual value function $\varphi(\cdot)$ and monotone allocation rule $y(\cdot)$,

$$\mathbb{E}[\varphi(q)y(q)] = \mathbb{E}[-y'(q)\Phi(q)].$$

Step by step, that is,

$$\begin{aligned}
\mathbb{E}[\varphi(q)y(q)] &= \int_0^1 \varphi(q)y(q)\,dq & q \sim U[0,1] \\
&= \Phi(1)y(1) - \Phi(0)y(0) - \int_0^1 y'(q)\Phi(q)\,dq \\
&= 0 + \mathbb{E}[-y'(q)\Phi(q)].
\end{aligned}$$

because $\Phi(1) = 1 \cdot v(1) = 0$ as $v(1) = 0$, and $\Phi(0) = 0 \cdot v(0) = 0$. Notice that the weakly decreasing monotonicity of the allocation rule $y(\cdot)$ implies the non-negativity of $-y'(q)$. With the left-hand side of equation as the expected virtual surplus, it is clear that a higher cumulative virtual value implies no lower expected virtual surplus. By definition of $\bar{\Phi}(\cdot)$ as the concave hull of $\Phi(\cdot)$, $\Phi(q) \leq \bar{\Phi}(q)$ and, therefore, for any monotone allocation rule, in expectation, the ironed virtual surplus is at least the virtual surplus, i.e., $\mathbb{E}[-y(q)\Phi(q)] \leq \mathbb{E}[-y(q)\bar{\Phi}(q)]$.

To see the equality under the assumption that $y'(q) = 0$ for all q where $\bar{\Phi}(q) > \Phi(q)$, rewrite the difference between the ironed virtual surplus and the virtual surplus via equation as,

$$\mathbb{E}[\bar{\varphi}(q)y(q)] - \mathbb{E}[\varphi(q)y(q)] = \mathbb{E}[-y'(q)(\bar{\Phi}(q) - \Phi(q))].$$

The assumption on y' implies the term inside the expectation on the right-hand side is zero $\forall q$. □

Multiple Bidders

Imagine we have three bidders competing in a revenue-optimal auction for a single item. They are as follows:

- Bidder 1 is uniform. $F_1(v) = \frac{v-1}{H-1}$ on $[1, H]$.
- Bidder 2 is exponential. $F_2(v) = 1 - e^{-v}$ for $v \in (1, \infty)$.
- Bidder 2 is exponential. $F_3(v) = 1 - e^{-2v}$ for $v \in (1, \infty)$.

What does the optimal mechanism look like?

First we calculate their virtual value functions.

- $f_1(v) = \frac{1}{H-1}$ for $v \in [1, H]$. $\varphi_1(v) = 2v - H$.
- $f_2(v) = e^{-v}$ for $v \in (1, \infty)$. $\varphi_2(v) = v - 1$.
- $f_3(v) = 2e^{-2v}$ for $v \in (1, \infty)$. $\varphi_3(v) = v - \frac{1}{2}$.

The bidders have personalized reserve prices (i.e., have positive virtual values with v_i above) $r_1 = \frac{H}{2}$, $r_2 = 1$, $r_3 = \frac{1}{2}$. Note that based on the support of F_2 and F_3 that bidder 2 and 3 are always above their reserve prices.

The optimal mechanism excludes bidder 1 if $v_1 < r_1 = \frac{H}{2}$, and otherwise allocates to the bidder with the largest virtual value $\varphi_i(v_i)$. If some $\varphi_j(v_j)$ is the second highest virtual value and exceeds its reserve price, then bidder i pays a price of $\varphi_i^{-1}(\varphi_j(v_j))$; otherwise, bidder i just pays r_i.

Definition 5. A *reserve price* r is a minimum price below which no buyer may be allocated the item. There may also be personalized reserve prices r_i where if $v_i < r_i$ then v_i will not be allocated to. Bidders above their reserves participate in the auction.