

Single-Parameter Optimal Revenue (continued)

Virtual Welfare Recap

- Maximize welfare ($\sum_i v_i x_i$): Always give the bidder the item, always give it away for free!
- Maximize revenue: Post a price that maximizes $REV = \max_r r \cdot [1 - F(r)]$.

Using only the revelation principle and the payment identity $p_i(b_i, \mathbf{b}_{-i}) = b_i \cdot x_i(b_i, \mathbf{b}_{-i}) - \int_0^{b_i} x_i(z, \mathbf{b}_{-i}) dz$, we proved the following:

$$\text{REVENUE} = \mathbb{E}_{\mathbf{v} \sim \mathbf{F}} \left[\sum_i p_i(\mathbf{v}) \right] = \mathbb{E}_{\mathbf{v} \sim \mathbf{F}} \left[\sum_i \varphi_i(v_i) x_i(\mathbf{v}) \right] = \text{VIRTUAL WELFARE}$$

where

$$\varphi_i(v_i) = v_i - \frac{[1 - F_i(v_i)]}{f_i(v_i)}.$$

Then similarly to welfare, just give the item to the bidder with the highest (non-negative) *virtual* value! But this doesn't work when $\varphi(\cdot)$ isn't monotone, because then $x(\cdot)$ wouldn't be.

Definition 1. A distribution F is regular if the corresponding virtual valuation function $\varphi(v) = v - \frac{1-F(v)}{f(v)}$ is strictly increasing.

Claim 1. A virtual welfare maximizing allocation x is monotone if and only if the virtual value functions are regular.

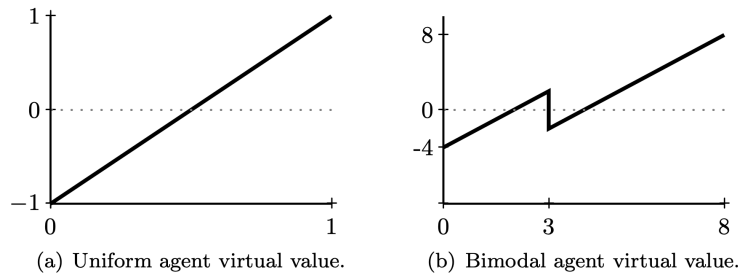


Figure 1: Virtual value functions $\varphi(v) = v - \frac{1-F(v)}{f(v)}$ for the uniform and bimodal agent examples.

It will be helpful to keep the following two examples in mind:

- a uniform agent with $v \sim U[0, 1]$. Then $F(x) = x$ and $f(x) = 1$. $\varphi(v) = 2v - 1$.

b. a bimodal agent with

$$v \sim \begin{cases} U[0, 3] & w.p. \frac{3}{4} \\ U(3, 8) & w.p. \frac{1}{4} \end{cases} \quad \text{and} \quad f(v) = \begin{cases} \frac{3}{4} & v \in [0, 3] \\ \frac{1}{20} & v \in (3, 8] \end{cases}$$

$$1 - F(v) = \begin{cases} \frac{1}{4} + \left(\frac{3-v}{3}\right) \cdot \frac{3}{4} & v \in [0, 3] \\ \left(\frac{8-v}{5}\right) \cdot \frac{1}{4} & v \in (3, 8] \end{cases} \quad \text{so} \quad \varphi(v) = \begin{cases} \frac{4}{3}(v-1) & v \in [0, 3] \\ 2v-8 & v \in (3, 8] \end{cases}$$

Quantile Space and Ironing

Instead of talking in *value space*, where an agent has value v , the fraction of the distribution with value above v is $1 - F(v)$, and the revenue from posting a “take-it-or-leave-it” price of v is $v[1 - F(v)]$, we will instead talk about *quantiles*.

Let $1 - F(v) = q$, the fraction of the distribution with a value at least v , willing to pay a price of v . *Quantile q* refers to the the fraction of the distribution left above its corresponding value. For example, consider a distribution that is $U[\$0, \$10]$. Then the quantile 0.1 corresponds to \$9, where 10% of the population might have a higher value. We let $v(q)$ denote the corresponding value, so $v(0.1)$ is \$9.

Definition 2. The *quantile* of a single-dimensional agent with value $v \sim F$ is the measure with respect to F of stronger values, i.e., $q = 1 - F(v)$; the inverse demand curve maps an agent’s quantile to her value, i.e., $v(q) = F^{-1}(1 - q)$.

Quantiles are particularly useful because we can draw an agent from any distribution by drawing a quantile $q \sim U[0, 1]$. That is, for any \hat{q} and any distribution F , $\Pr_F[q \leq \hat{q}] = \hat{q}$. In English: the probability that an agent has a value in the top 0.3 of the distribution is 0.3.

For everything we do today, we *could* stay in value space, but we’d have to normalize by the distribution using $f(v)$, which makes everything a bit messier and a bit trickier.

Example: For the example of a uniform agent where $F(z) = z$, the inverse demand curve is $v(q) = 1 - q$.

For an allocation rule $x(\cdot)$ in value space, we define an allocation rule in *quantile space* $y(\cdot)$:

$$y(q) = x(v(q)).$$

As $x(\cdot)$ is monotone weakly increasing, then $y(\cdot)$ is monotone *weakly decreasing*.

Definition 3. The *price-posting revenue curve* of a single-dimensional linear agent specified by inverse demand curve $v(\cdot)$:

Claim 2. Any allocation rule $y(\cdot)$ can be expressed as a distribution of posted prices.

Proof.

Claim 3. A distribution F is regular if and only if its corresponding price-posting revenue curve is concave.

Observe that $P'(q) = \varphi(v(q))$:

$$P'(q) = \frac{d}{dq} (q \cdot v(q)) = v(q) + qv'(q) = v - \frac{1 - F(v)}{f(v)} = \varphi(v(q)).$$

Thus $\Phi(q) = \int_0^q \varphi(\hat{q}) d\hat{q} = P(q)$.

Definition 4. The *ironing procedure* for (non-monotone) virtual value function φ (in quantile space) is:

(i) Define the cumulative virtual value function as

(ii) Define ironed cumulative virtual value function

(iii) Define the ironed virtual value function as

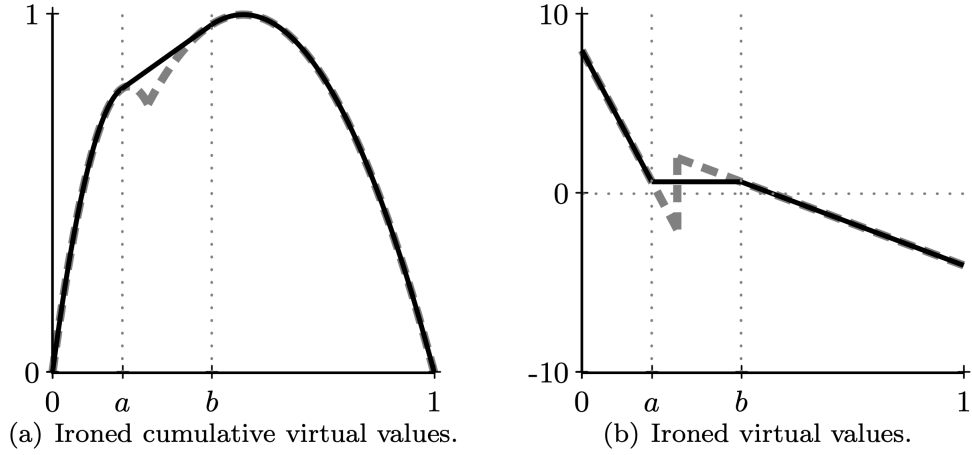


Figure 2: The bimodal agent's (ironed) revenue curve and virtual values in quantile space.

Theorem 1. For any monotone allocation rule $y(\cdot)$ and any virtual value function $\varphi(\cdot)$, the expected virtual surplus of an agent is upper-bounded by her expected ironed virtual surplus, i.e.,

$$\mathbb{E}[\varphi(q)y(q)] \leq \mathbb{E}[\bar{\varphi}(q)y(q)].$$

Furthermore, this inequality holds with equality if the allocation rule y satisfies $y'(q) = 0$ for all q where $\bar{\Phi}(q) > \Phi(q)$.

Proof.

Multiple Bidders

Imagine we have three bidders competing in a revenue-optimal auction for a single item. They are as follows:

- Bidder 1 is uniform. $F_1(v) = \frac{v-1}{H-1}$ on $[1, H]$.
- Bidder 2 is exponential. $F_2(v) = 1 - e^{-x}$ for $v \in (1, \infty)$.
- Bidder 3 is exponential. $F_3(v) = 1 - e^{-2x}$ for $v \in (1, \infty)$.

What does the optimal mechanism look like?

Definition 5. A *reserve price* r is a minimum price below which no buyer may be allocated the item. There may also be personalized reserve prices r_i where if $v_i < r_i$ then v_i will not be allocated to. Bidders above their reserves participate in the auction.