DS 574 Algorithmic Mechanism Design	**Lecture #4**
Fall 2022	Prof. Kira Goldner

Revenue Maximization and Myersonian Virtual Welfare

Bayesian Stages and Interim Rules

Using notions from the Bayesian setting and how bidders Bayesian update as they learn information, we define three stages of the auction:

1. *ex ante*: Before any information has been drawn; i only knows $\mathbf{F}$.
2. *interim*: Values v_i have been drawn; i only knows their own valuation, and thus the updated prior $\mathbf{F} |_{v_i}$.
3. *ex post*: The auction has run and concluded. All bidders know all $v_1, \ldots, v_n$.

Typically we discuss the *ex post* allocation and payment rules as a function of all of the values. However, in the Bayesian setting, to reason about BIC, it often makes sense to take in terms of *interim* allocation and payment rules which have the same information as bidder i before the auction is run.

Definition 1. We define the *interim* allocation and payment rules in expectation over the updated Bayesian prior given i's valuation:

$$x_i(v_i) = \Pr_{\mathbf{F}}[x_i(\mathbf{v}) = 1 \mid v_i] = \mathbb{E}_{\mathbf{F}}[x_i(\mathbf{v}) \mid v_i]$$

and

$$p_i(v_i) = \mathbb{E}_{\mathbf{F}}[p_i(\mathbf{v}) \mid v_i].$$

Our definition of Bayesian Incentive-Compatibility then follows:

Definition 2. A mechanism with *interim* allocation rule x and *interim* payment rule p is Bayesian Incentive-Compatible (BIC) if

$$v_i x_i(v_i) - p_i(v_i) \geq v_i x_i(z) - p_i(z) \quad \forall i, v_i, z.$$

Using these, we can more easily prove the BIC/BNE versions of Myerson's Lemma and the Revelation Principle.

Virtual Welfare

Imagine a single buyer will arrive with their private value v. We want to design DSIC mechanisms.

What mechanism should you use to maximize *welfare* ($\sum_i v_i x_i$) Always give the bidder the item, always give it away for free!

What should you do to maximize (expected) revenue? Post a price that maximizes $\textsc{Rev} = \max_r r \cdot [1 - F(r)]$.

Definition 3. In a deterministic mechanism, given other bids $\mathbf{b}_{-i}$, bidder i's *critical bid* is the minimum bid $b_i^* = \min\{b_i : x_i(b_i, \mathbf{b}_{-i}) = 1\}$ such that bidder i is allocated to.

Then with $\mathbf{b}_{-i}$ fixed, for all winning $v_i \geq b_i^*$, i's payment $p_i(v_i, \mathbf{b}_{-i}) = b_i^*$ is their critical bid.

What is winner i's critical bid in a single-item auction? The second-highest bid!

What about in the k identical item setting? The $k+1^{\text{st}}$ bid!

Maximizing Expected Revenue

Recall:

- The revelation principle says that it's without loss to focus only on truthful mechanisms.
- Payment is determined by the allocation:

$$p_i(b_i, \mathbf{b}_{-i}) = b_i \cdot x_i(b_i, \mathbf{b}_{-i}) - \int_0^{b_i} x_i(z, \mathbf{b}_{-i})\, dz$$

We want to maximize $\mathbb{E}_{\mathbf{v}\sim\mathbf{F}}[\sum_i p_i(\mathbf{v})]$.

$$\begin{aligned}
\mathbb{E}_{v_i\sim F_i}[p_i(v_i, \mathbf{v}_{-i})] &= \int_0^\infty f_i(v_i) p_i(v_i, \mathbf{v}_{-i})\, dv_i \\
&= \int_0^\infty f_i(v_i) \left[v_i \cdot x_i(v_i, \mathbf{v}_{-i}) - \int_0^{v_i} x_i(z, \mathbf{v}_{-i})\, dz \right] dv_i \\
&= \int_0^\infty \left[f_i(v_i) v_i x_i(v_i, \mathbf{v}_{-i}) - x_i(v_i, \mathbf{v}_{-i}) \left[\int_{v_i}^\infty f_i(z)\, dz \right] \right] dv_i \qquad (*) \\
&= \int_0^\infty \left[f_i(v_i) v_i x_i(v_i, \mathbf{v}_{-i}) - x_i(v_i, \mathbf{v}_{-i})[1 - F_i(v_i)] \right] dv_i \\
&= \int_0^\infty f_i(v_i) x_i(v_i, \mathbf{v}_{-i}) \left[v_i - \frac{[1 - F_i(v_i)]}{f_i(v_i)} \right] dv_i \\
&= \mathbb{E}_{v_i\sim F_i}[\varphi_i(v_i) x_i(v_i, \mathbf{v}_{-i})]
\end{aligned}$$

where

$$\varphi_i(v_i) = v_i - \frac{[1 - F_i(v_i)]}{f_i(v_i)}$$

is the Myersonian virtual value and $(*)$ follows by switching the order of integration. Then

$$\text{Revenue} = \mathbb{E}_{\mathbf{v}\sim\mathbf{F}}[\sum_i p_i(\mathbf{v})] = \sum_i \mathbb{E}_{\mathbf{v}\sim\mathbf{F}}[p_i(\mathbf{v})] = \sum_i \mathbb{E}_{\mathbf{v}\sim\mathbf{F}}[\varphi_i(v_i) x_i(v_i, \mathbf{v}_{-i})]$$

Note that this does require takes $\mathbb{E}_{\mathbf{v}_{-i}\sim\mathbf{F}_{-i}}$ of both sides of our previous equation.

$$= \mathbb{E}_{\mathbf{v}\sim\mathbf{F}}[\sum_i \varphi_i(v_i) x_i(\mathbf{v})] = \text{Virtual Welfare}$$

Given this conclusion, how should we design our allocation rule x to maximize expected virtual welfare (expected revenue)? Give the item to the bidder with the highest *virtual* value!

When would this cause a problem with incentive-compatibility? When the corresponding x isn't monotone!

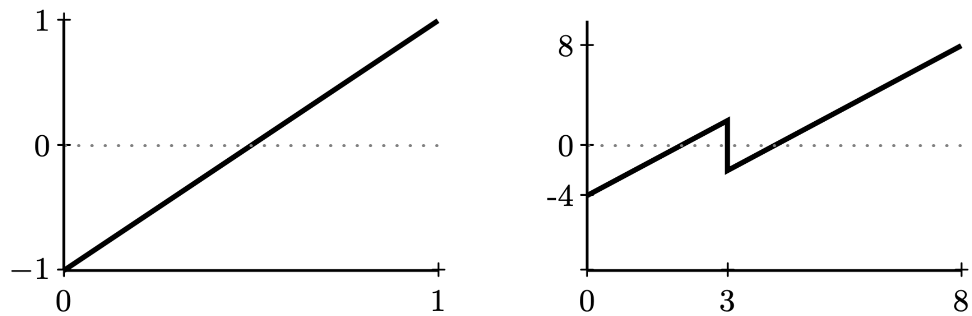
Definition 4. A distribution F is regular if the corresponding virtual valuation function $\varphi(v) = v - \frac{1-F(v)}{f(v)}$ is strictly increasing.

Suppose we are in the single-item setting and all of the distributions are regular. What do the payments look like in the virtual-welfare-maximizing allocation?

For a fixed $\mathbf{b}_{-i}$, if i is the winner, then i's payment is i's critical bid, which is $\varphi_i^{-1}(b_2)$ where b_2 is the second highest bid. Exercise: what about for k identical items?

Claim 1. A virtual welfare maximizing allocation x is monotone if and only if the virtual value functions are regular.

Exercise: Argue this.



(a) **Uniform agent virtual value.** (b) **Bimodal agent virtual value.**

Figure 1: Virtual value functions $\varphi(v) = v - \frac{1-F(v)}{f(v)}$ for the uniform and bimodal agent examples.

It will be helpful to keep the following two examples in mind:

a. a uniform agent with $v \sim U[0,1]$. Then $F(x) = x$ and $f(x) = 1$.

b. a bimodal agent with

$$v \sim \begin{cases} U[0,3] & w.p. \frac{3}{4} \\ U(3,8] & w.p. \frac{1}{4} \end{cases} \qquad \text{and} \qquad f(v) = \begin{cases} \frac{3}{4} & v \in [0,3] \\ \frac{1}{20} & v \in (3,8] \end{cases}$$

Do the following:

- Calculate the virtual values for both examples.

  a. $\varphi(v) = 2v - 1$

b. $1 - F(v) = \begin{cases} \frac{1}{4} + \left(\frac{3-v}{3}\right) \cdot \frac{3}{4} & v \in [0,3] \\ \left(\frac{8-v}{5}\right) \cdot \frac{1}{4} & v \in (3,8] \end{cases}$ so $\varphi(v) = \begin{cases} \frac{4}{3}(v-1) & v \in [0,3] \\ 2v - 8 & v \in (3,8] \end{cases}$

- Are they regular? Are there any issues using the allocation that maximizes expected virtual welfare?

  a. Yep!

  b. Nope. As we can see in Figure 1, $\varphi(3.5) = -1 < \varphi(2) = \frac{4}{3}$. This implies a bidder gets allocated with $v = 2$ but then stops getting allocated as they increase their value to 3.5.

- What does that allocation actually look like?

  a. Allocate to all bidders above $v = 0.5$ at a price (critical bid) of $\varphi^{-1}(0) = 0.5$.

  b. The virtual welfare maximizing allocation isn't DSIC! Turns out you can do something to make φ monotone and *then* use the VW-maximizing allocation. We'll do this later in class.