

## Revenue Maximization and Myersonian Virtual Welfare

### Bayesian Stages and Interim Rules

Using notions from the Bayesian setting and how bidders Bayesian update as they learn information, we define three stages of the auction:

1. *ex ante*: Before any information has been drawn;  $i$  only knows  $\mathbf{F}$ .
2. *interim*: Values  $v_i$  have been drawn;  $i$  only knows their own valuation, and thus the updated prior  $\mathbf{F} | v_i$ .
3. *ex post*: The auction has run and concluded. All bidders know all  $v_1, \dots, v_n$ .

Typically we discuss the *ex post* allocation and payment rules as a function of all of the values. However, in the Bayesian setting, to reason about BIC, it often makes sense to take in terms of *interim* allocation and payment rules which have the same information as bidder  $i$  before the auction is run.

**Definition 1.** We define the *interim* allocation and payment rules in expectation over the updated Bayesian prior given  $i$ 's valuation:

$$x_i(v_i) = \Pr_{\mathbf{F}}[x_i(\mathbf{v}) = 1 | v_i] = \mathbb{E}_{\mathbf{F}}[x_i(\mathbf{v}) | v_i]$$

and

$$p_i(v_i) = \mathbb{E}_{\mathbf{F}}[p_i(\mathbf{v}) | v_i].$$

Our definition of Bayesian Incentive-Compatibility then follows:

**Definition 2.** A mechanism with *interim* allocation rule  $x$  and *interim* payment rule  $p$  is Bayesian Incentive-Compatible (BIC) if

$$v_i x_i(v_i) - p_i(v_i) \geq v_i x_i(z) - p_i(z) \quad \forall i, v_i, z.$$

### Virtual Welfare

Imagine a single buyer will arrive with their private value  $v$ . We want to design DSIC mechanisms.

What mechanism should you use to maximize *welfare*?

What should you do to maximize (expected) revenue?

**Definition 3.** In a deterministic mechanism, given other bids  $\mathbf{b}_{-i}$ , bidder  $i$ 's *critical bid* is the minimum bid  $b_i^* = \min\{b_i : x_i(b_i, \mathbf{b}_{-i}) = 1\}$  such that bidder  $i$  is allocated to.

Then with  $\mathbf{b}_{-i}$  fixed, for all winning  $v_i \geq b_i^*$ ,  $i$ 's payment  $p_i(v_i, \mathbf{b}_{-i}) = b_i^*$  is their critical bid.

What is winner  $i$ 's critical bid in a single-item auction?

What about in the  $k$  identical item setting?

## Maximizing Expected Revenue

Recall:

- The revelation principle says that it's without loss to focus only on truthful mechanisms.
- Payment is determined by the allocation:

$$p_i(b_i, \mathbf{b}_{-i}) = b_i \cdot x_i(b_i, \mathbf{b}_{-i}) - \int_0^{b_i} x_i(z, \mathbf{b}_{-i}) dz$$

We want to maximize  $\mathbb{E}_{\mathbf{v} \sim \mathbf{F}}[\sum_i p_i(\mathbf{v})]$ .

$$\mathbb{E}_{v_i \sim F_i}[p_i(v_i, \mathbf{v}_{-i})] =$$

where

$$\varphi_i(v_i) =$$

is the Myersonian virtual value and (\*) follows by switching the order of integration. Then

$$\text{REVENUE} = \mathbb{E}_{\mathbf{v} \sim \mathbf{F}}[\sum_i p_i(\mathbf{v})] =$$

= VIRTUAL WELFARE

Given this conclusion, how should we design our allocation rule  $x$  to maximize expected virtual welfare (expected revenue)?

When would this cause a problem with incentive-compatibility?

**Definition 4.** A distribution  $F$  is regular if the corresponding virtual valuation function  $\varphi(v) = v - \frac{1-F(v)}{f(v)}$  is strictly increasing.

Suppose we are in the single-item setting and all of the distributions are regular. What do the payments look like in the virtual-welfare-maximizing allocation?

For a fixed  $\mathbf{b}_{-i}$ , if  $i$  is the winner, then  $i$ 's payment is  $i$ 's critical bid, which is

Exercise: what about for  $k$  identical items?

**Claim 1.** A virtual welfare maximizing allocation  $x$  is monotone if and only if the virtual value functions are regular.

Exercise: Argue this.

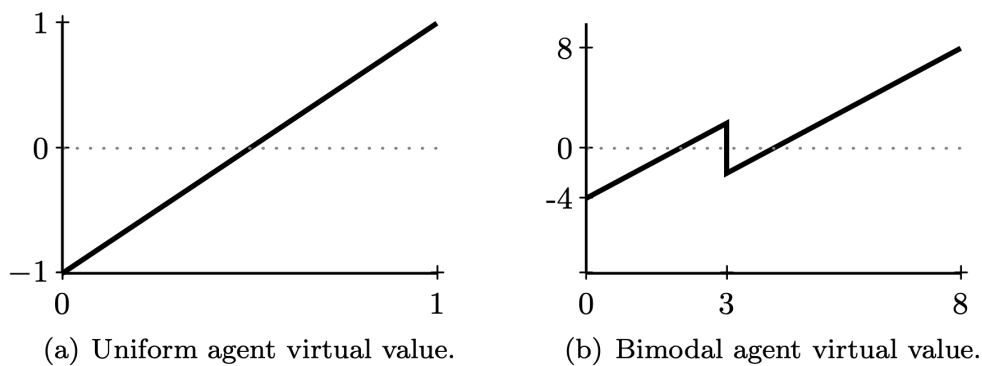


Figure 1: Virtual value functions  $\varphi(v) = v - \frac{1-F(v)}{f(v)}$  for the uniform and bimodal agent examples.

It will be helpful to keep the following two examples in mind:

a. a uniform agent with  $v \sim U[0, 1]$ . Then  $F(x) = x$  and  $f(x) = 1$ .

b. a bimodal agent with

$$v \sim \begin{cases} U[0, 3] & w.p. \frac{3}{4} \\ U(3, 8) & w.p. \frac{1}{4} \end{cases} \quad \text{and} \quad f(v) = \begin{cases} \frac{3}{4} & v \in [0, 3] \\ \frac{1}{20} & v \in (3, 8] \end{cases}$$

Do the following:

- Calculate the virtual values for both examples.
- Are they regular? Are there any issues using the allocation that maximizes expected virtual welfare?
- What does that allocation actually look like?