

Recap

Mechanisms (1) receive bids, (2) determine who gets the good, and (3) decide on payments. The question is, how should they do (2) and (3)?

When we design a mechanism, we aim for the following three desirable properties:

- (1) **Strong incentive guarantees.** e.g., Dominant Strategy Incentive-Compatibility (DSIC) and Individual Rationality (IR).
- (2) **Strong performance guarantees.** e.g., maximizing social welfare (or surplus) $\sum_{i=1}^n v_i x_i$.
- (3) **Computational efficiency.** e.g., implementation in polytime.

In general, as we design mechanisms, we'll take the following design approach:

Step 1: Assume, without justification, that bidders bid truthfully. Then, how should we assign bidders to slots so that properties (2) strong performance guarantees and (3) computational efficiency hold?

Step 2: Given our answer to Step 1, how should we set selling prices so that property (1) strong incentive guarantees holds?

Terms we've discussed and/or defined so far:

- Incentive-compatibility / truthfulness
 - Dominant Strategy Incentive-Compatibility (DSIC)
 - Individual Rationality (IR)
 - Best-response
 - Equilibrium
 - Allocation rule x
 - Payment rule p
 - Quasilinear utility $u_i(\mathbf{b})$
 - Feasible outcomes X
- Auction types
 - First-price auction
 - Vickrey auction / Second-price auction
 - All-pay auction
 - Single-parameter environments
 - single-item
 - k identical items
 - sponsored search
- Formal definitions
 - Strategy of bids \mathbf{b} with bids from all bidders except i , \mathbf{b}_{-i}
 - Myerson's Lemma
 - implementable mechanisms
 - monotone allocation rules
 - the payment identity

The Revelation Principle

So far, we've been investigating *Dominant-Strategy Incentive-Compatible (DSIC)* mechanisms. To be DSIC, this means that

- (1) Every participant in the mechanism has a dominant strategy, no matter what their private valuation is.
- (2) This dominant strategy is *direct revelation*, where the participant truthfully reports all of their private information to the mechanism.

There are mechanisms that satisfy (1) but not (2). To give a silly example, imagine a single-item auction in which the seller, given bids \mathbf{b} , runs a Vickrey auction on the bids $3\mathbf{b}$. Every bidder's dominant strategy is then to bid $v_i/3$.

For a formal definition of a direct revelation mechanism:

Definition 1. A mechanism is *direct revelation* if it is single-round, sealed-bid, and has action space equal to the type (value) space. That is, an agent can bid any type they might have, and an agent's action *is* bidding a type.

The Revelation Principle and the Irrelevance of Truthfulness

The Revelation Principle states that, given requirement (1), there is no need to relax requirement (2): it comes “for free.”

Theorem 1 (Revelation Principle for DSIC Mechanisms). *For every mechanism M in which every participant has a dominant strategy (no matter what their private information), there is an equivalent direct-revelation DSIC mechanism M' .*

Equivalent here means that as a function of the *valuation profile* (not bids), the allocation and payment $(x(\mathbf{v}), p(\mathbf{v}))$ are equivalent in both M and M' .

Proof. The proof uses a simulation argument; see Figure 1. By assumption, in mechanism M , every bidder i has a dominant strategy $\sigma_i(v_i)$ whatever their v_i .

We construct the following mechanism M' , the mechanism takes over the responsibility of applying the dominant strategy. Precisely, (direct-revelation) mechanism M' accepts sealed bids b_1, \dots, b_n from the players. It submits the bids $\sigma_1(b_1), \dots, \sigma_n(b_n)$ to the mechanism M , and chooses the same outcome (e.g., winners of an auction and selling prices) that M does.

Mechanism M' is DSIC: If a participant i has private information v_i , then submitting a bid other than v_i can only result in M' playing a strategy other than $\sigma_i(v_i)$ in M , which can only decrease i 's utility. \square

The takeaway from the Revelation Principle (Theorem 1) is that **it is without loss to design direct revelation mechanisms**. That is, you might as well require your mechanism to be **incentive-compatible**.

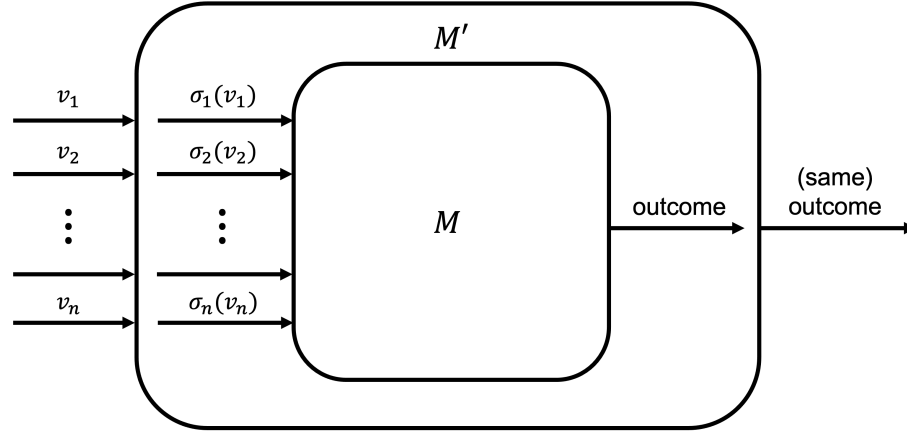


Figure 1: Proof of the Revelation Principle. Construction of the direct-revelation mechanism M' , given a mechanism M with dominant strategies.

Beyond Dominant-Strategy: Bayesian Settings

There are many reasons why we can't always require dominant strategies when design mechanisms.

- (1) Requiring such a strong concept might not be tractable.
- (2) Agents do not always have dominant strategies! What then?

We'll now introduce the Bayesian setting.

Suppose the valuation v_i of bidder i is drawn from a prior distribution F_i .

- We abuse notation and let F_i denote the cumulative distribution function (CDF) of the distribution; that is, $F_i(x) = \Pr_{v_i \sim F_i}[v_i \leq x]$.
- We use $f_i(x)$ to denote the probability density function (pdf) of the distribution; that is, $f_i(x) = \frac{d}{dx} F_i(x)$.
- We use \mathbf{F} or \vec{F} to denote the *joint distribution* of the marginal buyer distributions F_i . That is, if the buyers are independently distributed, then \mathbf{F} is the product distribution $\mathbf{F} = \times_i F_i$. (Note however that buyers are not necessarily independently distributed in all settings.)

Unless otherwise noted, we assume that the prior distribution \mathbf{F} is *common knowledge* to all bidders and the mechanism designer (the seller).

Definition 2. A *Bayes-Nash equilibrium (BNE)* for a joint distribution \mathbf{F} is a strategy profile $\sigma = (\sigma_1, \dots, \sigma_n)$ such that for all i and v , $\sigma_i(v_i)$ is a best-response when other agents play $\sigma_{-i}(\mathbf{v}_{-i})$ when $\mathbf{v}_{-i} \sim \mathbf{F}_{-i} | v_i$.

Claim 1. Consider two identically and independently drawn bidders from $F = U[0, 1]$. It is a (symmetric) BNE for each bidder to bid $\sigma_i(v_i) = v_i/2$ in the first-price auction.

Proof. Suppose bidder 2 is playing the strategy $\sigma_2(v_2) = v_2/2$. We prove that bidder 1's best-response is $\sigma_1(v_1) = v_1/2$. For now, call the bid given by $\sigma_1(v_1) = b_1$.

Given bidder 2's strategy, bidder 1's expected utility is

$$\begin{aligned}
 \mathbb{E}_{v_2}[u_i(\sigma_1(v_1), \sigma_2(v_2))] &= v_1 \cdot \mathbb{E}_{v_2}[x_1(\sigma_1(v_1), \sigma_2(v_2))] - \mathbb{E}_{v_2}[p_1(\sigma_1(v_1), \sigma_2(v_2))] \\
 &= v_1 \cdot \Pr_{v_2}[b_1 > v_2/2] - \mathbb{E}_{v_2}[b_1 \mid b_1 > v_2/2] && x, p \text{ in FPA} \\
 &= \Pr_{v_2}[b_1 > v_2/2] \cdot [v_1 - b_1] && \text{def } \mathbb{E} \\
 &= F(2b_1) \cdot [v_1 - b_1] && \text{def } F(\cdot) \\
 &= (2b_1) \cdot [v_1 - b_1] && F = U[0, 1] \\
 &= 2b_1v_1 - 2b_1^2 \\
 \frac{d}{db_1} \mathbb{E}_{v_2}[u_i(b_1, v_2/2)] &= 2v_1 - 4b_1 && \text{differentiate to max} \\
 \implies b_1 &= v_1/2
 \end{aligned}$$

Hence bidder 1's best-response strategy is to bid $\sigma_1(v_1) = v_1/2$ in response to $\sigma_2(v_2) = v_2/2$, and thus these strategies are a BNE. \square

Theorem 2 (Revenue Equivalence). *The payment rule and revenue of a mechanism is uniquely determined by its allocation. Hence, any two mechanisms with the same allocation must earn the same revenue.*

What is this theorem a corollary of? Prove this for the first-price auction and the Vickrey (second-price) auction in the above setting!

Proof. This is just a corollary of Myerson's Lemma! As we pointed out, the *only* variables in the payment identity are the allocation rule! Payment is 100% determined by the allocation rule! Then two mechanisms with the same allocation *must* have the same payments.

Consider the first-price auction and the second-price auction each with two bidders i.i.d. from $U[0, 1]$. Let V^1 and V^2 denote the random variables that are the highest and second-highest draws from $U[0, 1]$, respectively. Note that two draws from the uniform distribution evenly divide the interval in expectation: $\mathbb{E}_{V^1, V^2 \sim U[0, 1]}[V^1] = 2/3$ and $\mathbb{E}_{V^1, V^2 \sim U[0, 1]}[V^2] = 1/3$.

In the first-price auction, the item is allocated to V^1 at a payment of its BNE bid of $V^1/2$. Then the expected winner's payment (and thus revenue) is $\frac{1}{2}\mathbb{E}[V^1] = 1/3$.

In the second-price auction, the item is allocated to V^1 at a payment of the second-highest bid $b^2 = V^2$, since Vickrey is DSIC. Then the expected winner's payment (and thus revenue) is $\mathbb{E}[V^2] = 1/3$. \square

Bayesian Settings

Using notions from the Bayesian setting and how bidders Bayesian update as they learn information, we define three stages of the auction:

1. *ex ante*: Before any information has been drawn; i only knows \mathbf{F} .
2. *interim*: Values v_i have been drawn; i only knows their own valuation, and thus the updated prior $\mathbf{F} \mid v_i$.

3. *ex post*: The auction has run and concluded. All bidders know all v_1, \dots, v_n .

Typically we discuss the *ex post* allocation and payment rules as a function of all of the values. However, in the Bayesian setting, to reason about BIC, it often makes sense to take in terms of *interim* allocation and payment rules which have the same information as bidder i before the auction is run.

Definition 3. We define the *interim* allocation and payment rules in expectation over the updated Bayesian prior given i 's valuation:

$$x_i(v_i) = \Pr_{\mathbf{F}}[x_i(\mathbf{v}) = 1 \mid v_i] = \mathbb{E}_{\mathbf{F}}[x_i(\mathbf{v}) \mid v_i]$$

and

$$p_i(v_i) = \mathbb{E}_{\mathbf{F}}[p_i(\mathbf{v}) \mid v_i].$$

Our definition of Bayesian Incentive-Compatibility then follows:

Definition 4. A mechanism with *interim* allocation rule x and *interim* payment rule p is Bayesian Incentive-Compatible (BIC) if

$$v_i x_i(v_i) - p_i(v_i) \geq v_i x_i(z) - p_i(z) \quad \forall i, v_i, z.$$

Exercise (optional):

- Extend Myerson's Lemma and the payment identity for Bayesian Incentive-Compatible (BIC) mechanisms.
- Extend the Revelation Principle for BIC mechanisms.