DS 574 Algorithmic Mechanism Design	**Lecture #16**
Fall 2022	Prof. Kira Goldner

The Bulow-Klemperer Result

One famous result takes the form of resource augmentation.

Theorem 1 (Bulow Klemperer '96). *For i.i.d. regular single-item environments, the expected revenue of the second-price auction with* $n+1$ *agents is at least that of the optimal auction with* n *agents.*

Let's talk about what this theorem is saying. Instead of finding the optimal auction tailored to a distribution F for n agents, you can use the Vickrey auction, which requires no prior knowledge of the distribution, so long as we require one extra bidder, regardless of the n that we start with, and earn more revenue than optimal. We do have two strong assumptions here (aside from being in the single-item environment):

- Bidders are i.i.d.—every bidder's value is drawn from F, and independently at that.
- F is a regular distribution. That is, $v - \frac{1-F(v)}{f(v)}$ is monotone non-decreasing.

This result **does not hold** without these assumptions. However, it is a *very strong* result, should our setting meet these assumptions.

Proof. First we claim that in the i.i.d. setting, the Vickrey auction earns the most revenue of all mechanism that *must allocate the item*. To maximize expected revenue, we know it is equivalent to maximize virtual welfare. If we *must* allocate the item in every case, then we should allocate the item to the bidder with the highest virtual value *even* when the virtual value is negative. Because we are in the i.i.d. setting *and* F is regular so $\varphi(\cdot)$ is monotone, virtual value functions are identical, so the bidder with the highest virtual value is identical to the bidder with the highest value. That is, the allocation rule is to *always* allocate to the highest bidder. This is precisely the allocation rule of the Vickrey auction.

Now, we compare the revenue of the Vickrey auction on $n+1$ bidders to another auction that always allocates the item, and there earns at most as much revenue—call this mechanism M. This mechanism runs the revenue-optimal mechanism for n bidders on the first n bidders $1, \ldots, n$. If the item is not allocated in that mechanism, it is then allocated to bidder $n+1$, so the item is always allocated, and is designed for $n+1$ bidders.

Then clearly

$$\textsc{opt}(n, F) \leq \textsc{Rev}_M(n+1, F) \leq \textsc{Rev}_{\text{Vickrey}}(n+1, F).$$

□

For more recent and complex Bulow-Klemperer style or "competition complexity" results, some examples include [4, 1, 5, 2].

The Single Sample Mechanism

Can't recruit extra buyers? Instead, we can just exclude one. This is what the single sample result says.

Theorem 2. *Given a random sample from a bidder's distribution, posting it as a take-it-or-leave-it price gives a $\frac{1}{2}$-approximation to the optimal revenue.*

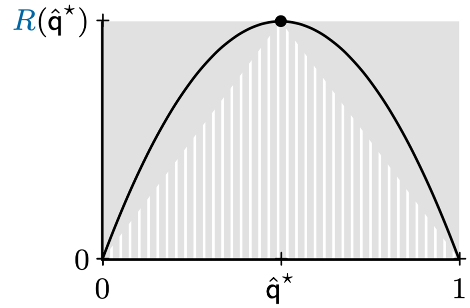

Figure 1: Geometric intuition for a posted-price from a single sample.

Proof. In quantile space! A randomly sampled value corresponds to a randomly sampled *quantile*, sampled uniformly $q \sim U[0,1]$ independent of the bidder's distribution. The revenue from this mechanism (call it M) is precisely $\text{Rev}(M) = \mathbb{E}_{q\sim U[0,1]}[R(q)] = \int_0^1 R(q)\, dq$ where R is the price-posting revenue curve in quantile space. This is exactly the area under the $R(\cdot)$.

What does depend on the bidder's distribution is the *optimal* quantile to sell to at a posted price, some q^*. The optimal single-bidder revenue that we aim to approximate is $\text{OPT} = R(q^*)$. This is exactly the area of the rectangle with a height of $R(q^*)$ (the highest height of the curve) and the full width of the curve from 0 to 1 (a width of 1)—$R(q^*) \cdot 1$.

Now notice that the area under the curve contains the triangle with corners at $(0,0)$, $(0,1)$ and $(R(q^*),1)$. Hence this triangle must have area $R(q^*)/2$, that is, $\text{OPT}/2$, contained in the area under of the curve, which is equal to

$$\text{Rev}(M) \geq \text{OPT}/2.$$

□

It turns out, using a single sample from the buyers' distribution to set reserve prices and running VCG is a good approximation to the optimal mechanism. See Hartline chapter 5 for more.

Interested in these sort of sample complexity results? A good foundational result is [7], and [3] then [8] after that. A more recent result that also contains an introduction surveying other results is [6].

Prophet Inequalities

See notes from Lecture 15.

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