DS 574 Algorithmic Mechanism Design	**Lecture #11 Worksheet**
Fall 2022	Prof. Kira Goldner

Last Time: Intro to LP Duality

Primal (P):

$$\begin{aligned} \max \quad & \mathbf{c}^T\mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq 0 \end{aligned}$$

Dual (D):

$$\begin{aligned} \min \quad & \mathbf{y}^T\mathbf{b} \\ \text{subject to} \quad & \mathbf{A}^T\mathbf{y} \geq \mathbf{c} \\ & \mathbf{y} \geq 0 \end{aligned}$$

The dual program is defined precisely so that weak duality holds: any feasible dual solution $y \in (D)$ provides an upper bound to any feasible primal solution $x \in (P)$ (and vice versa). The dual of the dual is the primal.

Theorem 1 (Weak Duality)**.** *If* $\mathbf{x}$ *is feasible in (P) and* $\mathbf{y}$ *is feasible in (D) then* $\mathbf{c}^T\mathbf{x} \leq \mathbf{b}^T\mathbf{y}$.

Proof.

$$\mathbf{c}^T\mathbf{x} \overset{1}{\leq} (\mathbf{A}^T\mathbf{y})\mathbf{x} = \mathbf{y}^T\mathbf{A}\mathbf{x} \overset{2}{\leq} \mathbf{y}^T\mathbf{b} = \mathbf{b}^T\mathbf{y}.$$

Where (1) follows by the dual constraints $\mathbf{A}^T\mathbf{y} \geq \mathbf{c}$ and (2) follows by the primal constraints $\mathbf{A}\mathbf{x} \leq \mathbf{b}$. □

What is not trivial (or by definition) is *strong duality*, and in fact, it is so involved that we will not even prove the hard direction: that an optimal solution always exists.

Conditions for Optimality

Strong Duality

Theorem 2 (Strong Duality)**.** *A pair of solutions* $(\mathbf{x}^*, \mathbf{y}^*)$ *are optimal for the primal and dual respectively if and only if* $\mathbf{c}^T\mathbf{x}^* = \mathbf{b}^T\mathbf{y}^*$.

Proof. $(\Rightarrow)$ Skip.
$(\Leftarrow)$

Complementary Slackness

Primal (P):

$$\begin{aligned} \max \quad & \mathbf{c}^T\mathbf{x} \\ \text{subject to} \quad & \sum_i a_{ji}x_i \le b_j \quad \forall j \quad (y_j) \\ & x_i \ge 0 \quad \forall i \end{aligned}$$

Dual (D):

$$\begin{aligned} \min \quad & \mathbf{y}^T\mathbf{b} \\ \text{subject to} \quad & \sum_i a_{ij}y_i \ge c_i \quad \forall i \quad (x_i) \\ & y_j \ge 0 \quad \forall j \end{aligned}$$

Theorem 3 (Complementary Slackness)**.** *A pair of solutions* $(\mathbf{x}^*, \mathbf{y}^*)$ *are optimal for the primal and dual respectively if and only if the following complementary slackness conditions (1) and (2) hold:*

Proof.

Maximizing Welfare in the Unit Demand Setting

Given n unit-demand bidders and m items, determine the allocation rule that maximizes welfare. Do this by formulating a linear program.

Determine your objective, decision variables, and constraints.

Formulate the dual.

Do you see an interpretation of this dual?

Separation Oracles

Fact 1 (Ellipsoid Algorithm)**.** Every linear program that admits a polynomial-time separation oracle can be solved in polynomial time.

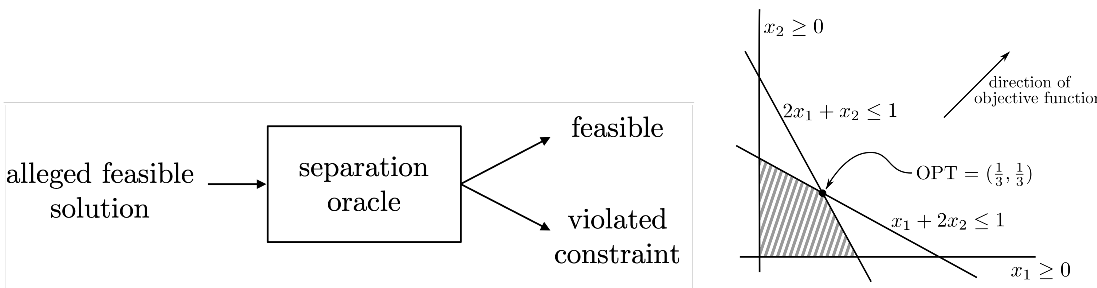

Figure 1: Left: A sketch of a separation oracle. For example, in the toy example on the right, on the alleged feasible solution $(\frac{1}{3}, \frac{1}{2})$, the separation oracle may return the violated constraint $x_1 + 2x_2 \leq 1$.

Consider a linear program such that:

a. There are n decision variables.

b. There are any number of constraints, for example, exponential in n. These constraints are not provided explicitly as input.

c. There is a polynomial-time *separation oracle* for the set of constraints. By "polynomial-time," we mean running time polynomial in n and the maximum number of bits of precision required. A separation oracle (Figure 1) is a subroutine that takes as input an alleged feasible solution to the LP, and either (i) correctly declares the solution to be feasible, or (ii) correctly declares the solution to be infeasible, and more strongly provides a proof of infeasibility in the form of a constraint that the proposed solution violates.

(The ellipsoid algorithm is not actually practical, but there are other algorithms that *are* often practically useful that rely on a separation oracle, such as cutting plane methods.)

Revenue Maximization

Consider the single-item revenue-maximization setting with n bidders. Formulate the LP to maximize revenue. Formulate its dual.