DS 320 Algorithms for Data Science	**Logic & Notation**
Spring 2024	Prof. Kira Goldner

This guide is based on content by Prof. Krzysztof Onak from September 2023.

1 Basic Statements

When you do any type of mathematical reasoning, you are bound to make basic statements about objects of your interest. Examples of such statements include "$x < y + 1$," "k is even," and "7 divides $t + 3$." Depending on the specific values of variables involved in such statements, they can be either true or false. However, you will often want to combine such statements into more complex statements. In this document, the plan is to go over some logical operators that will let you do it.

2 Conjunctions and Disjunctions

Conjunction. Suppose that you have two logical statements P and Q. If you write "P and Q" or "$P \wedge Q$," the new statement you just created is true if and only if *both* P and Q are true. This kind of statement is called the conjunction of P and Q.

Disjunction. If you write "P or Q" or "$P \vee Q$," the new statement you just created is true if at least one of P and Q is true. If neither P or Q is true, the new statement is false. This kind of statement is called the disjunction of P and Q.

3 Quantifiers

What if you want to write a conjunction or disjunction of a large number of similar statements. This is where you may want to start using phrases such as "for all" or "there exists." For instance, if you want to define what it means for a natural number to be prime, you could express it as

$$k \text{ is a prime} \equiv k > 1 \text{ and for all } t \in \{2, \ldots, k-1\},\ t \text{ does not divide } k$$

This is exactly what mathematical quantifiers are for and we will now discuss the two most popular ones.

3.1 For All/For Every/For Any

We start by discussing the *universal quantifier*. In this case, we have some logical statement $P(x)$ that involves a variable x. Now we want to say that for all, or for every, or for any setting of x, the statement is true. We can also specify a subset to which x belongs. For example, we can say

For every even integer x greater than 2, x is composite.

In this case, $P(x) =$ "x is composite". This kind of statement is true if $P(x)$ is true for all valid settings of x. It's probably obvious that the above example is true. To show that a specific statement involving this type of quantifier is true, you have to show that $P(x)$ is true for all settings of x. To show that it's false, it suffices to demonstrate a single x for which $P(x)$ is false. An example of a false statement is "For all $x \in \mathbb{N}$, x is prime."

Instead of writing "for all," you can sometimes use a math symbol for this: $\forall$ (`\forall` in LaTeX). Hence you can write

$$\forall x \in \{4, 6, 8, \ldots\} . (x \text{ is composite})$$

to rewrite the previous statement.

3.2 There Exists/For Some

We now discuss the *existential quantifier*. In this case, we again have some logical statement $P(x)$ involving a variable x, and this time we want to express the fact that there is a setting of x that makes $P(x)$ true. This corresponds to phrases such as "for some x ..." or "there exists an x such that ..." in English. For instance, we can say

$$\text{There exists a } x \in \mathbb{Z} \text{ such that } x^2 = 100.$$

In this case, $P(x) =$ "$x^2 = 100$". This kind of statement is true if there is at least one setting of x that makes $P(x)$ true. Hence to prove this kind of statement, it suffices to show a single x that makes it true. To prove that it's false, you have to show that $P(x)$ is false for all values of x.

Once again there is a math symbol for this: $\exists$ (`\exists` in LaTeX). We can therefore rewrite the previous statement as

$$\exists x \in \mathbb{Z} . x^2 = 100$$

and you may also be able to notice that this specific statement is true.

3.3 Quantifiers vs. Natural Languages

Sentences in any natural language can be very difficult to interpret so you have to be careful what you write. For instance when you say "there is an x such that $P(x, y)$ for all y" this could be interpreted as either "$\exists x \forall y . P(x, y)$" or "$\forall y \exists x . P(x, y)$." These two phrases can have a very different meaning. From my experience putting quantifiers first usually makes things more clear in this kind of setting. So if you write

"for all y, there is an x such that $P(x, y)$"

or

"there is an x such that for all y, $P(x, y)$"

the ordering of quantifiers will be easier I think.

4 Sample Problems Demonstrating a Proof Involving the Universal Quantifier

4.1 Find a Large Number

Problem: Find a number $x \in \mathbb{R}$ such that for any real $a > 0$, $x > 15 - a$.

Solution: This is of course a very simple problem. The set of correct solutions includes 15, 15.0001, 320, 2023, and $(100\pi)^e$ but the point here is that when you define your solution there is no a to which you can refer. As an example, let us select $x = 15.0001$ as our solution. Now we have to show that for all $a \in (0, \infty)$, $x > 15 - a$. Consider any $a > 0$. We have $15 - a < 15 < 15.0001 = x$, which proves that x has the desired properties. □

4.2 Example 2: Superpolynomial Function

Problem: Design a function $f : \mathbb{N} \to \mathbb{R}$ such that for all $a > 0$, $n^a = O(f(n))$.

Solution: We pick $f(n) = 2^n$. Let us now prove that it has the desired properties, which requires proving that for all $a \in (0, \infty)$, $n^a = O(f(n))$. To this end, consider any specific $a \in (0, \infty)$. Note that $\lim_{n\to\infty} \frac{n^a}{2^n} = \lim_{n\to\infty} 2^{a \log_2 n - n} = 0$ because $\lim_{n\to\infty}(a \log_2 n - n) = -\infty$. Hence, there is some constant c such that for $n \geq c$, $2^n \geq n^a$. This proves that $n^a = O(f(n))$—as desired—and finishes our proof. □

5 Bonus: Useful Mathematical Notation

5.1 Sets of Numbers

- $\mathbb{R}$ — real numbers
- $\mathbb{Z}$ — integers (all of them, both positive, negative, and zero)
- $\mathbb{N} = \{0, 1, 2, \ldots\}$ — natural numbers
- $\mathbb{Z}_+ = \{i \in \mathbb{Z} : i > 0\}$ — positive integers
- $\mathbb{Z}_- = \{i \in \mathbb{Z} : i < 0\}$ — positive integers

5.2 Floor and Ceiling

Floor and ceiling are useful operators that convert real numbers to the closest integers.

Floor. The floor operator, $\lfloor \cdot \rfloor$, rounds real numbers down to the closest integer. Formally, it can be defined as

$$\lfloor x \rfloor = \max\{i \in \mathbb{Z} : i \leq x\}.$$

Ceiling. The ceiling operator, $\lceil \cdot \rceil$, rounds real numbers up to the closest integer. Formally, it can be defined as

$$\lceil x \rceil = \min\{i \in \mathbb{Z} : i \geq x\}.$$