DS 320 Algorithms for Data Science — **Lecture #6**
Spring 2024 — Prof. Kira Goldner

Introduction to Greedy: Shortest Path

Say we want to find the shortest path from home to any other point in the city. How might we do that? One way would be to use BFS.

But, what if there's congestion on the roads? That is, what if the graph is weighted?

Definition 1. Let w_e (or w_{uv}) denote the *weight* of edge e (or (u,v)).

We can think of this as the length of the edge, or the time (or cost) to traverse it.

Question: Given a graph G, how can we find the shortest (least-weight) path from s to any other vertex v?

Algorithm 1 Dijkstra's Algorithm(G, w)

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Input: Graph $G = (V, E)$ and weights $w$.
Let $S$ be the set of explored nodes
for each $u \in S$, store a distance $d(u)$
Initially $S = \{s\}$ and $d(s) = 0$
while $S \neq V$ do
    let $d'(v) = \min_{(u,v) \in E, u \in S} d(u) + w_{uv}$
    select $v \in \text{argmin}_{v \notin S} d'(v)$
    add $v$ to $S$ and set $d(v) = d'(v)$
end while
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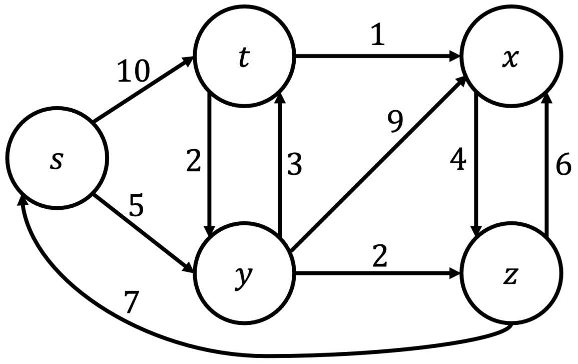

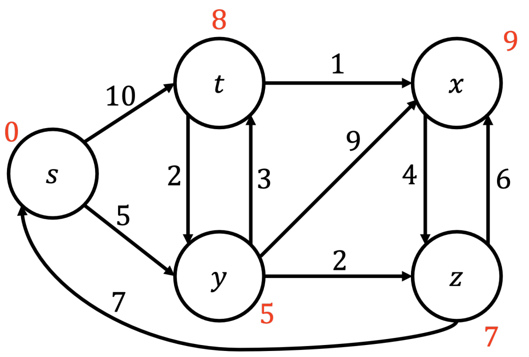

Figure 1: A weighted graph G (left) and the shortest path from s calculated for each vertex (right).

Let P_v denote the shortest path to v from s—that is, $P_v = P_u \cup (u,v)$ when $v \in \text{argmin}_{v \notin S} d'(v)$ and $(u,v) \in \text{argmin}_{(u,v) \in E, u \in S} d(u) + w_{u,v}$.

Greedy Stays Ahead

There are four main steps for a greedy stays ahead proof.

Step 1: Define your solutions. Describe the form your greedy solution takes, and what form some other solution takes (possibly the optimal solution). For example, let A be the solution constructed by the greedy algorithm, and let O be a (possibly optimal) solution.

Step 2: Find a measure. Find a *measure* by which greedy stays ahead of the other solution you chose to compare with. Let $a_1, \dots, a_k$ be the first k measures of the greedy algorithm, and let $o_1, \dots, o_m$ be the first m measures of the other solution ($m = k$ sometimes).

Step 3: Prove greedy stays ahead. Show that the partial solutions constructed by greedy are always just as good as the initial segments of your other solution, based on the measure you selected.

- For all indices $r \leq \min(k, m)$, prove (often by induction) that $a_r \geq o_r$ or that $a_r \leq o_r$, whichever the case may be. Don't forget to use your algorithm to help you argue the inductive step.

Step 4: Prove optimality. Prove that since greedy stays ahead of the other solution with respect to the measure you selected, then it is optimal.

Proof of Dijkstra

Then we prove the following via a "greedy stays ahead"-style induction.

Lemma 1. *Consider the set S at any point in the algorithm's execution. For each $u \in S$, P_u is a shortest $s - u$ path.*

(Proof of Correctness: When the algorithm terminates, S contains all nodes, thus by Lemma 1, we'll have found shortest paths from s to all nodes.)

Proof. By induction on $|S|$.

Base case ($|S| = 1$): $|S| = 1$, so $S = \{s\}$ and $d(s) = 0$.

Inductive Hypothesis: Suppose the claim holds for some $k \geq 1$.

Inductive Step ($|S| = k + 1$): The algorithm grows to $|S| = k + 1$ by adding some node v. Let (u, v) be the final edge on P_v. By the IH, P_u is the shortest $s - u$ path for all $u \in S$.

Consider any other $s - v$ path P. It must leave S somewhere—call (x, y) the edge that first crosses. But P cannot be shorter than P_v, as it contains P_x and (x, y), and Dijkstra could have just added y to S with this P_y if it were shortest, but instead it chose $P_u + (u, v)$ as shorter. $\square$

Runtime. A basic analysis gives $O(nm)$, since there are n iterations of growing S and we search at most m edges each time. An implementation with a clever data structure—a heap-based priority queue—improves this to $O(m \log n)$.

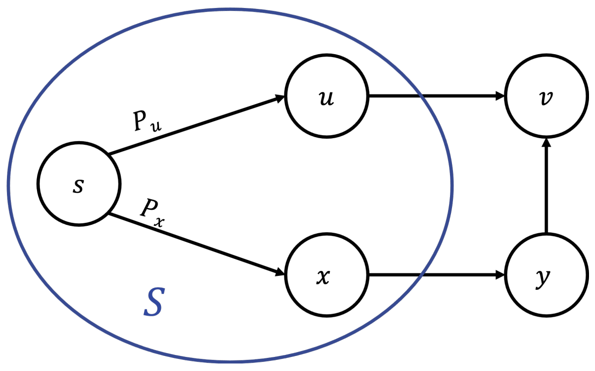

Figure 2: Illustration of the argument in the inductive step.

Observe. Dijkstra is really a continuous/waterfilling-version of BFS.

Greedy II: Interval Scheduling

Suppose you are given n jobs to schedule on a machine. Each job i (where $i \in \{1, \ldots, n\}$) has a start time $s(i)$ and a finish time $f(i)$. You would like to schedule *as many* jobs as possible given that the machine can only process one job at a time, and the jobs must run from their start time to finish time uninterrupted to be processed. That is, the machine cannot process two jobs that overlap.

What *greedy* algorithm should you use to schedule the jobs? By what metric is it greedy? (See **Step 2**.)

To be continued next lecture...