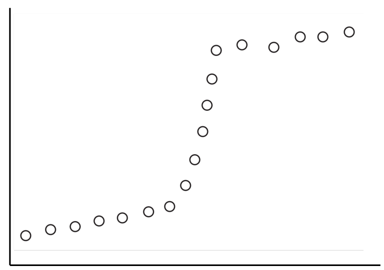
DS 320 Algorithms for Data Science | **Lecture #15**
Spring 2024 | Prof. Kira Goldner

Dynamic Programming III: Segmented Least Squares

The Problem

We are given a set of points $\{p_1 = (x_1, y_2), p_2 = (x_2, y_2), \dots, p_n = (x_n, y_n)\}$ sorted by x-coordinate. Our goal is to fit a (segmented) line to P with least squares error.



What is "error" here? We use square error (SSE) from any line we use. That is, if our line is determined by slope a and y-intercept b, then our SSE would be

$$SSE = \sum_{i=1}^{n} (y_i - ax_i - b)^2.$$

Using calculus, we can derive that this is minimized when we set

$$a = \frac{n \sum_i x_i y_i - (\sum_i x_i)(\sum_i y_i)}{n \sum_i x_i^2 - (\sum_i x_i)^2} \qquad \text{and} \qquad b = \frac{\sum_i y_i - a \sum_i x_i}{n}.$$

But what if we can use as many segments as we want, just with a penalty c for each additional segment? How should we decide on the number of segments, and on what the segments should look like?

Our goal is to partition P into some C contiguous segments with minimal least squares error when there is a penalty c for each segment.

Making the Key Observation

The last point p_n belongs to a single segment which must begin somewhere. Where does it begin? In each case, what does the optimal solution look like?

Step 1: The Subproblem

Let $\text{OPT}(i)$ denote the optimum solution for the points $p_1, \dots, p_i$, and $\text{OPT}(0) = 0$. Let $e_{j,i}$ denote the minimum error of any line with respect to points $p_j, \dots, p_i$ (i.e., find the line using the calculus solution up above).

Step 2: The Recurrence

If the last segment of the optimal partition is $p_i, \dots, p_n$, then: $\textsc{opt}(n) = e_{i,n} + c + \textsc{opt}(i-1)$.

Hence

$$\textsc{opt}(i) = \min_{1 \le j \le i} \{e_{j,i} + c + \textsc{opt}(j-1)\}.$$

where we use the segment $p_j, \dots, p_i$ if and only if $j \in$ argmin of the above.

Step 3: Prove that your recurrence is correct. In the optimal solution on i points, i must be in a segment that starts at the $j \in$ argmin of the above. $\textsc{opt}(j-1)$ gives the optimal segmented SSE for the first $i-1$ points and $e_{j,i}$ gives the optimal SSE for the segment from j to i, so adding these two error terms plus the penalty of c for using the one additional segment from j to i is the valid cost of this solution. If i instead was in a different segment that started at a different j', then for the same reasons, the cost of this solution would be $\textsc{opt}(j'-1) + c + e_{j',i}$, but this term did not minimize the above which is why it was not selected, hence it cannot have optimal (minimal) error. Hence the above recurrence is correct.

Step 4: State and prove your base cases. $\textsc{opt}(0) = 0$ is enough to get us off the ground. Notice that $\textsc{opt}(1)$ is then well-defined.

Step 5: State how to solve the original problem. This is once again $\textsc{opt}(n)$.

Step 6: The Algorithm

Algorithm 1 SegmentedLeastSquares($p_1, \dots, p_n$)

Input: Set of n points $p_1, \dots p_n$.
Initialize memo array M of length $n+1$ with $M[0] = 0$.
for all pairs $j \le i$ **do**
  Compute $e_{j,i}$ for the segment $p_j, \dots, p_i$
end for
for $i = 1, \dots, n$ **do**
  $M[i] = \min_{1 \le j \le i}\{e_{j,i} + c + M[j-1]\}$
end for
return $M[n]$

Step 7: Running Time

The recurrence optimizes over n things, and we loop over n entries, so filling the memo takes $O(n^2)$ time. Solving for $e_{j,i}$ takes $O(n)$ time and there are $O(n^2)$ pairs of (j, i), so this takes $O(n^3)$ time, which is the dominant term for the algorithm.

Algorithm 2 FindSegments(i)

Input: Number of points n.

Initialize memo array M of length $n+1$ with $M[0] = 0$.

if $i = 0$ **then**

  return $Null$

else

  Find an j that minimizes $e_{j,i} + c + M[j-1]$

  return the segment $\{p_j, \ldots, p_i\}$ and FindSegments($i-1$)

end if