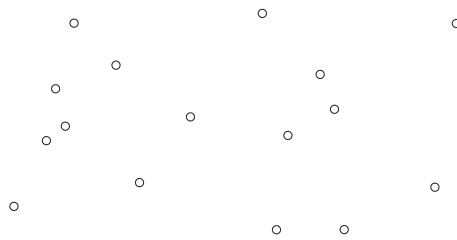


## Divide & Conquer II: Closest Pair of Points

### The Problem

Your input for the *closest pair of points* problem is a set  $P$  of  $n$  points in  $\mathbb{R}^2$ . Assume that all of the  $x$  and  $y$  coordinates are distinct. The goal is to output a pair of points  $p_1$  and  $p_2$  minimizing the Euclidean  $L_2$  distance  $d(p_1, p_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ .



This problem extremely common: in graphics, computer vision, robotics, scientific simulation, etc.

As we mentioned last time, Divide & Conquer is usually the way to come up with a *more efficient* algorithm for a problem which already has a polynomial brute force solution. What's the **naïve algorithm** here and what is its **running time**?

Just compare all pairs of points:  $O(n^2)$ .

### Step 1: Define your recursive subproblem.

*Hint:* An idea similar to mergesort works here.

In preprocessing, sort  $P$  by both  $x$  and  $y$  coordinate. Call these  $P_x$  and  $P_y$ . Divide the points in half by  $x$  coordinate, where the first (left) half is  $L$  and the second (right) half is  $R$ .

The dividing line  $h$  will go through the rightmost point in  $L$ .

### Step 2: Combine the solutions to your subproblems.

Given the solutions (closest pair) from your subproblems (make sure the parameters make sense), how do you combine or compare these solutions to get the solution to our problem of size  $n$ ?

The closest pair will be one of: (1) the closest pair from  $L$ , (2) the closest pair from  $R$ , or (3) the closest pair across the halves. The solution to our subproblems will return the first two, but how do we find the solution to (3) in order to compare these?

## Computing the Closest Pair Across Halves

*Idea:* Narrow down the set of points we need to search.

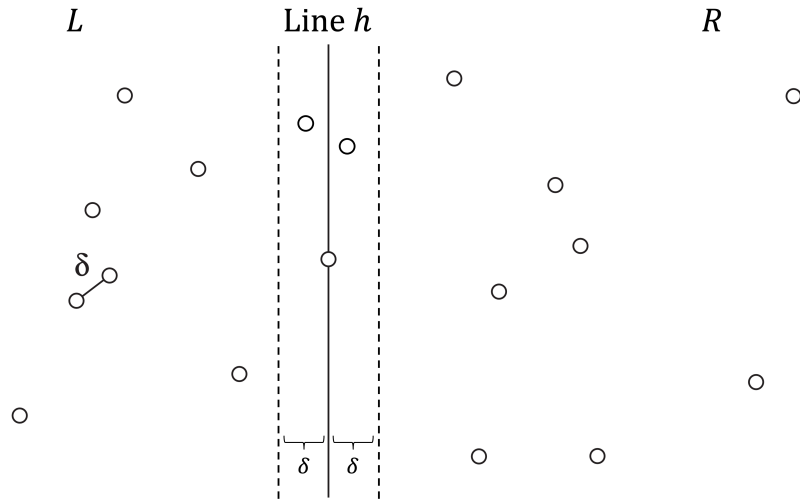
**Lemma 1.** For subproblem solutions of closest pairs  $\ell_0^*, \ell_1^* \in L$  and  $r_0^*, r_1^* \in R$ , let

$$\delta = \min\{ d(\ell_0^*, \ell_1^*) , d(r_0^*, r_1^*) \}.$$

If  $\ell \in L$  and  $r \in R$  and  $d(\ell, r) \leq \delta$ , then  $\ell$  and  $r$  are within  $\delta$  of  $h$ .

*Proof.*

Suppose some  $\ell \in L$  and  $r \in R$  and  $d(\ell, r) \leq \delta$ . Then the distance in the  $x$ -coordinates  $|x_\ell - x_r|$  must also be less than  $\delta$ . If we call  $x_h$  the  $x$ -coordinate of the line, then by definition of  $h$ ,  $x_\ell \leq x_h < x_r$ , hence it must also be the case that these distances  $|x_\ell - x_h|$  and  $|x_h - x_r|$  are at most  $\delta$ , that is, that  $r$  and  $\ell$  fall within  $\delta$  of the line  $h$ .



**Definition 1.** Let  $S = \{p \in P : d(p, h) < \delta\}$  be the set of points within distance  $\delta$  of the line  $h$ .

Note: We can compute  $S$  in linear time.

Lemma 1 implies that we only need to search  $S$ . Problem: Brute-force over  $S$  is  $O(n^2)$  still.

*Idea:* Brute force more intelligently.

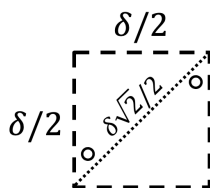
**Lemma 2.** Let  $S_y$  be a list of the points in  $S$  sorted by  $y$ -coordinate and let  $(s, s')$  be closest pair of points in  $P$ . If  $s, s' \in S$ , then they cannot lie more than 15 positions apart in  $S_y$ .

(There are less than  $15n$  such pairs, hence linear time to brute force!)

*Proof.*

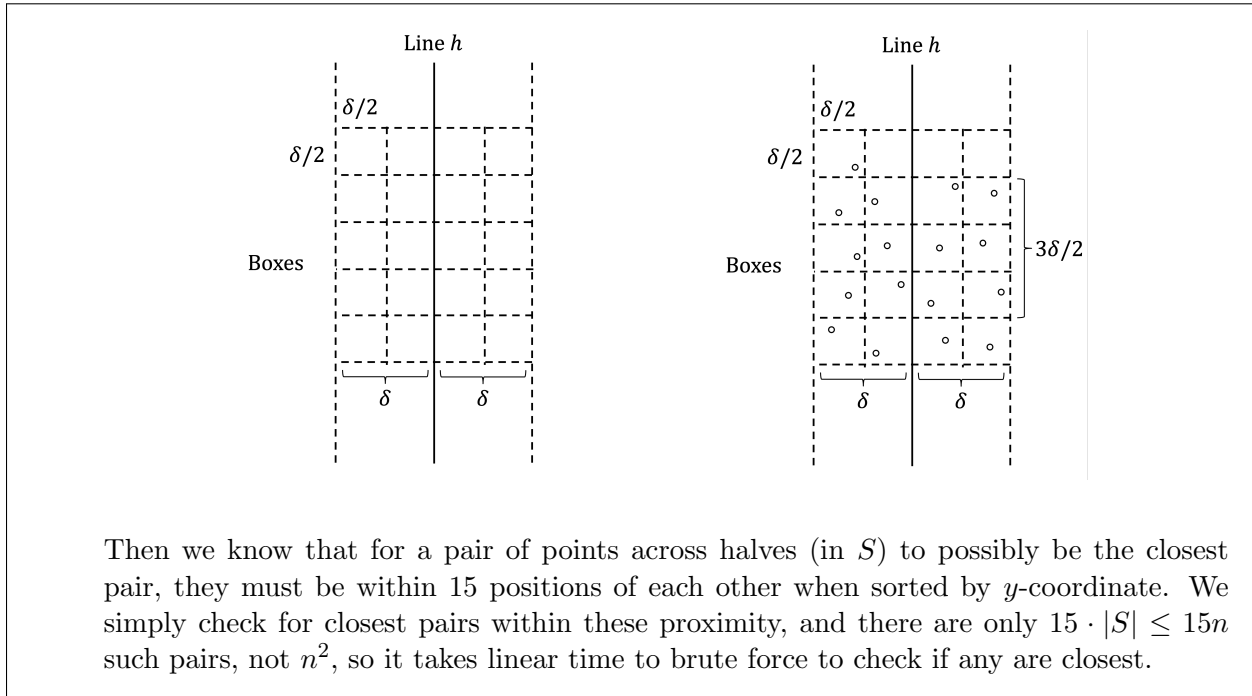
- a. Divide  $S$  into boxes of side-length  $\delta/2$ .
- b. **Show:** No two points can share a box.

If they did, they're on the same side and they're closer than  $\delta$  apart, which is a contradiction to what we know about the closest pairs in  $L$  and  $R$ .



- c. **Show:** If  $s$  and  $s'$  are at least 16 positions apart in  $S_y$ , then they're not the closest pair of points.

If  $s$  and  $s'$  are at least 16 positions apart in the list  $S_y$  (which is the list of points in  $S$ , the points within  $\delta$ -distance of  $h$ , sorted by  $y$ -coordinate), then they must be at least 3 rows of boxes apart. By part (b) we know that there cannot be more than one point in a box, so even if the points are packed as closely as possible, they must each lie in their own box, and thus two points that are 15 positions apart by  $y$ -coordinate, even when most densely packed, must have at least three full rows of boxes between them. Then the distance between these points  $d(s, s')$  must be at least  $3\delta/2 > \delta$ , and hence these points are not closer than the closest pair in one of the halves.



## The Algorithm

Preprocessing:

$P_x = P$  sorted by  $x$  coord.

$P_y = P$  sorted by  $y$  coord.

$\text{Closest}(P_x, P_y)$

$\text{Closest}(P_x, P_y)$ :

Base case.  $|P| \leq 3$ : Brute force

Construct  $L_x, L_y, R_x, R_y$

$(\ell_0^*, \ell_1^*) = \text{Closest}(L_x, L_y)$

$(r_0^*, r_1^*) = \text{Closest}(R_x, R_y)$

Construct  $S_y$

$(s_0^*, s_1^*) = \text{closest pair in } S_y \text{ within 15 spots of each other.}$

Return closest of  $(\ell_0^*, \ell_1^*), (r_0^*, r_1^*), (s_0^*, s_1^*)$

## Runtime

$$T(n) = aT(n/b) + O(f(n))$$

$$a =$$

$$b =$$

$$f(n) =$$

$$\Rightarrow T(n) =$$

$$T(n) = 2T(n/2) + O(n)$$

$$\Rightarrow T(n) = n \log n.$$