NP Completeness and Reductions

The 3-SAT Problem

Given a logical formula of n *boolean variables* $x_1, \dots, x_n$ put together using only conjunctions, disjunctions, and nots, determine if the formula can be satisfied.

Example:

$$\phi(x_1, x_2, x_3) = \quad (x_1 \vee \overline{x}_2) \wedge (\overline{x}_1 \vee \overline{x}_3) \wedge (x_2 \vee \overline{x}_3).$$

Is this formula ϕ satisfiable? Is there a satisfying assignment?

Yes: $x_1 = T$, $x_2 = T$, $x_3 = F$.

Definition 1. A specific instance of a variable x_i (negated or not) in the formula is referred to as a *literal*.

Example:

$$\phi = \quad x_1 \wedge x_2 \wedge \overline{x}_3 \wedge \overline{x}_1.$$

How many variables? 3. How many literals? 4.

Is ϕ satisfiable? No. $x_1 \wedge \overline{x}_1$ is unsatisfiable.

Definition 2. A formula is in *Conjunctive Normal Form (CNF)* if it can be broken down into clauses $C_1, \dots, C_m$ such that:

- Each clause C_i is the disjunction (OR) of literals.
- The formula is $C_1 \wedge \dots \wedge C_m$, the conjunction (AND) of clauses.

In k-SAT (e.g., 3-SAT), each clause contains k literals.

Example:

$$\phi = \quad (x_1 \vee x_2 \vee x_3) \wedge (\bar{x}_2 \vee \bar{x}_3 \vee \bar{x}_4) \wedge (\bar{x}_1 \vee \bar{x}_2 \vee x_4).$$

Give two satisfying assignments.

Facts about 3-SAT:

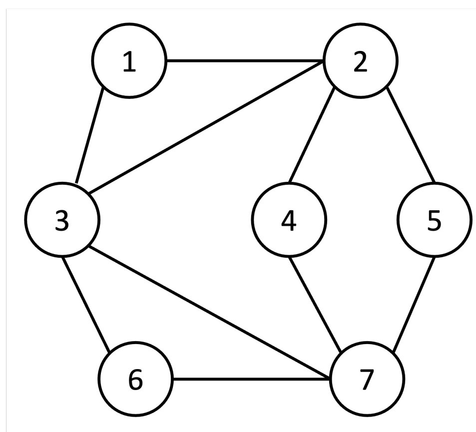
- Can represent any formula in 3-CNF
- Useful for artificial intelligence, circuit design, automatic theorem proving.
- Nobody knows how to solve it in polynomial time.
- Nobody knows how to prove that you can't, either.

Scary thing: There are thousands of problems like it.

Independent Set

Definition 3. A set of vertices $S \subseteq V$ is an *independent set* if for all $u, v \in S$, $(u, v) \notin E$.

Give an undirected graph $G = (V, E)$, output an independent set of maximum size.



The maximum size independent set is of size 4: $\{1, 4, 5, 6\}$.

3-SAT $\leq_P$ Independent Set

Our goal is to show that given an algorithm to solve Independent Set (a "black box"), we could then solve 3-SAT. Hence, Independent Set is in some sense *harder* than 3-SAT.

We will do this by

a. taking an instance of 3-SAT

b. from it, constructing an instance for Independent Set

c. showing why a solution to the Independent Set problem on this instance gives us back a solution for the 3-SAT instance in polynomial time.

This is called a polynomial-time reduction.

Our reduction: Given a boolean 3-CNF formula ϕ with n variables, m clauses each of 3 literals, we construct the following instance of independent set:

- Construct a vertex for each literal.

  If $v \in S$, the independent set, set the corresponding literal to True.
- Problem: What if S includes conflicting literals?

  Solution: Add an edge between conflicting literals.

  Interpretation: We're encoding truth "consistency" in our assignment.
- Last step: connect all vertices within a clause. (Why?)

  Consequence: Every IS has ≤ 1 vertex per clause triangle.

What's the largest that an independent set S could be for this construction? m.

Claim 1. ϕ is satisfiable if and only if $\exists$ and independent set S of size m.

($\Leftarrow$) Assume you have an independent set S of size m.

S must contain one vertex per clause gadget. For each $v \in S$, set the corresponding literals to True (and set any unset variables arbitrarily).

- Every clause is satisfied: one literal is true in each.
- This truth assignment is consistent: no conflicting literals.

($\Rightarrow$) Assume you have a satisfying assignment for ϕ.

Each clause is satisfied by at least one literal. Take one literal per clause, and add its vertex to the independent set S. This has size m (one per clause), and is independent:

- No conflicting literals.
- Never more than one literal per clause.

Algorithm for 3-SAT:

- Construct an instance of Independent Set (a graph).
- Call a black box algorithm for IS, finding a max IS S.
- Return SAT if $|S| = m$, UNSAT otherwise.

P vs. NP

Decision Problems

Definition 4. A *decision problem* is an algorithmic problem where the desired output is either *yes* or *no*.

Examples:

- 3-SAT
- Independent Set: Is the max IS of size $\geq k$?
- Knapsack: Is the set of value $\geq k$?
- Shortest Path: Is the path from s to t of length $\leq k$?

P: "Easy to Solve"

Definition 5. P (polynomial time) is a complexity class of decision problems. A decision problem is in P if there is an algorithm which solves it in polynomial time.

What are some problems in P?

NP: "Easy to Check"

Idea: If I claim to you that there's an independent set in G with $\geq k$ vertices, how could I convince you of that fact quickly?

What about for 3-SAT?

For all these problems, there's some information I can give you that you can use to quickly check that there's a yes instance. We call this information a *certificate* and the process you use to check correctness *certification*.

Formally: Given a decision problem A, a certifier algorithm ALG for A:

- Takes an input X for A and a "certificate" C.
  ("Here's a yes input, and here's proof that it's a yes input.")
- Returns True or False.
  ("Yep! Sure enough.")
- For every input X:
  (X is a yes instance) $\Leftrightarrow$ (There's some certificate C such that ALG returns true on (X, C).)
  - Can always find a certificate to convince me of a yes input.
  - Can't convince me that no inputs are actually yes inputs.

Examples:

- Independent Set (Is there an independent set of size $\geq k$?)
  - **Certificate:** The set.
  - **Certifier:** Check that set has size k, is independent.
- Knapsack (Is there a set of value at least k with weight at most W?)
  - **Certificate:** The set.
  - **Certifier:** Check that its value $\geq k$ and weight is $\leq W$.
- 3-SAT (Is ϕ satisfiable?)
  - **Certificate:** Variable assignment.
  - **Certifier:** Check each clause for satisfaction.

Definition 6. NP (non-deterministic polynomial time) is a complexity class of decision problems. A decision problem is in NP if there exists a polynomial-time certifier algorithm (that takes polynomial-size certificates).

$\Rightarrow$ all of the above problems are in NP.

We will only care about verifying "yes" instances. We don't care if it's hard to convince us that there's *no* independent set of size $\geq k$.

Not every problem is obviously NP: "Is this 3-SAT instance unsatisfiable?"

P vs. NP

Claim 1: $P \subseteq NP$

Ex. Weighted Interval Scheduling:

- I give you an instance of WIS, and claim that there's a schedule with weight $\geq k$.
- What certificate do I need to give you for you confirm this in polytime?
- Can just find WIS yourself!

To certify problems in P:

- Throw away the certificate.
- Solve the problem yourself.

Question: Does P = NP? (Or is there a problem in NP with provably no poly-time algorithm?)

Theorem 1 (Cook-Levin '71)**.** *For any problem $A \in NP$, $A \leq_P$ 3-SAT.*

Corollary 2. *If there's a polytime algorithm for 3-SAT, $P = NP$.*

Definition 7. A problem B is *NP-hard* if for all $A \in NP$, $A \leq_P B$.

Definition 8. A problem is *NP-complete* if it is NP-hard and also in NP.

Proving a problem is NP-Complete:

- Prove it's NP-hard (reduce from a known NP-hard problem)
- Prove it's in NP. (describe the certification algorithm)