DS 320 Algorithms for Data Science	**Lecture #16 Worksheet**
Spring 2023	Prof. Kira Goldner

Dynamic Programming IV: Shortest Paths

The Problem

Given a graph $G = (V, E)$ with *any edge weights* w_e for all $e \in E$, a source s, output the shortest $s - v$ path for every $v \in V$.

Naïve Approach

Let $N(v)$ be the neighbors of v.

- Subproblem: Let $\text{OPT}(v)$ denote the length of the shortest path from s to v.
- Recurrence: $\text{OPT}(v) = \min_{u \in N(v)} \text{OPT}(u) + w_{(u,v)}$.
- Base Case: $\text{OPT}(s) = 0$.

What's wrong with this approach? List any issues you see:

The Bellman-Ford Algorithm

Main Idea: Impose a measure of progress—parametrize the subproblems.

More specifically:

- Consider a shortest path from $s \to u \to v \to t$ with k edges
- $s \to u \to v$ is a shortest $s \to v$ path with $k - 1$ edges

Measure of progress: the number of edges in path. First compute all shortest paths with ≤ 1 edges. Then shortest paths with ≤ 2 edges. And so on, until we compute shortest paths of length $\leq n - 1$.

Structural Observation:

Proof.

Formal Description

State Your Subproblem:

State Your Recurrence:

Prove Your Recurrence:

State Your Base Cases:

Present Your Algorithm:

Runtime:

- Size of table:
- Time to fill in table:

  (How many table lookups?)

Total time:

Better Runtime Analysis

- How many row updates?
- How many table lookups per row update?
- Runtime =

Detecting Negative Cycles

Idea: Add an extra column (Column n) to the memo table.

Claim 1. There is a negative cycle in the graph if and only if Column $n \neq$ Column $n-1$.

Proof. ($\Leftarrow$) Suppose the graph has no negative cycles. Then by the Structural Observation,

($\Rightarrow$) Suppose there is a negative cycle in the graph.

To detect negative cycles:

- Run Bellman-Ford, add an nth column.
- If Column n = Column $n-1$, return "No Negative Cycles"
- Otherwise, return that there's a negative cycle.

Space-Efficient Bellman-Ford

How much space does Bellman-Ford use?

Observation:

Improved algorithm:

Total space usage: