**DS 320 Algorithms for Data Science** **Lecture #10**
**Spring 2023** Prof. Kira Goldner

# Divide & Conquer I: MergeSort

## The Problem

As with any sorting problem, our goal is to take in a list and return it sorted in non-decreasing order.

The main idea here is to break the list into two halves, and sort both halves. They we combine the two sorted halves together by continuing to take the smallest element. Each half is sorted recursively.

This is the general idea with all Divide & Conquer solutions. Typically, the "naive" algorithm is already a polynomial solution (like $\Theta(n^2)$ sorting algorithms). By identifying clever **subproblems** that we can divide our problem into, **solve recursively**, and **combine** our solutions together, we can give a *more efficient* algorithm.

## Divide & Conquer Outline

There are four main steps for a divide and conquer solution.

**Step 1: Define your recursive subproblem.** Describe in English what your subproblem means, what its parameters are, and anything else necessary to understand it.

Given an array $A$, the first index of a list `lo` and last index of a list `hi`, our recursive subproblem will be dividing the list in half at $\texttt{mid} = \lfloor(\texttt{lo} + \texttt{hi})/2\rfloor$, and then sorting each subproblem $\texttt{A}[\texttt{lo} \cdots \texttt{mid}]$ and $\texttt{A}[\texttt{mid} + 1 \cdots \texttt{hi}]$.

**Step 2: Define your base cases.** Your recursive algorithm has base cases, and you should state what they are.

Our base case will be when $\texttt{lo} = \texttt{hi}$ and hence $|A| = 1$, because a 1-element list is trivially sorted.

**Step 3: Present your recursive cases.** Give a mathematical definition of your subproblem in terms of "smaller" subproblems. Make sure your recursive call uses the same number and type of parameters as in your original definition.

$mergesort(A, \texttt{lo}, \texttt{lo}) = merge(\quad mergesort(A, \texttt{lo}, \texttt{mid}), \quad mergesort(A, \texttt{mid} + 1, \texttt{hi}) \quad )$ where $\texttt{mid} = \lfloor(\texttt{lo} + \texttt{hi})/2\rfloor$ and $merge(B, C)$ is a process of shuffling two sorted lists into one sorted list by taking and removing the smaller first from either.

We define a recursive algorithm that, given a list $A$ of elements as well as left and right indices `lo` and `hi`, returns the elements `A[lo]`, ..., `A[hi]` in non-decreasing sorted order.

**Algorithm 1** mergesort(A, lo, hi ).

```
Input: Array of elements A, left and right indices lo and hi.
if lo = hi then                                   // Base Case: |A| = 1
    return A[lo]
end if
mid = ⌊(lo + hi)/2⌋
L = mergesort(A, lo, mid)                         // Subproblem: Recursive calls on halves
R = mergesort(A, mid + 1, hi)
while both L and R are non-empty do               // Merge L and R into a single list S in Θ(n)
    frontL = L[1] and frontR = R[1]
    if frontL ≤ frontR then
        append frontL to S and remove it from L
    else
        append frontR to S and remove it from R
    end if
end while
if one of L or R is non-empty then
    append remaining list onto S
end if
return S
```

**Step 4: Prove correctness.** This will be an inductive proof that your algorithm does what it is supposed to. You should start by arguing that your base cases return the correct result, then for the inductive step, argue why your recursive cases combine to solve the overall problem.

**Claim 1.** `mergesort(L, lo, hi)` correctly sorts `A`$[$`lo` $\cdots$ `hi`$]$ in non-decreasing order.

*Proof.* For a list `A`$[$`lo` $\cdots$ `hi`$]$, we prove that `mergesort(A, lo, hi)` correctly sorts `A`$[$`lo` $\cdots$ `hi`$]$ into non-decreasing order for any list `A`$[$`lo` $\cdots$ `hi`$]$ by strong induction on $|A| =$ `hi` $-$ `lo` $+ 1$.

As a base case, consider when $|A| = 1$, i.e. when `hi` $=$ `lo`. This one-element list is already sorted, and our algorithm correctly returns `A[lo]` as the sorted list.

For the inductive hypothesis, suppose that the claim is true for *all* lists of length $< n$; that is, for any list `A` and `hi` $-$ `lo` $+ 1 < n$, `mergesort(A, lo, hi)` correctly sorts `A`$[$`lo` $\cdots$ `hi`$]$.

Now consider a list $A$ of length $n$. Our algorithm divides $A$ into two halves of size $< n$; therefore, $L$ and $R$ are in sorted order by our inductive hypothesis. The minimum element of $A$ is therefore either the minimum element of $L$ (which is at the front of $L$) or the minimum element of $R$ (which is at the front of $R$). We correctly take whichever is smallest as the minimum of $A$. We do this repeatedly, always selecting the next minimum element from the front of $L$ or $R$, until we've produced the sorted $A$.

Thus we have by induction that `mergesort(A, lo, hi)` correctly sorts any list `A`$[$`lo` $\cdots$ `hi`$]$. □

**Step 5: Prove running time.** This is especially important for divide and conquer solutions, as there is usually an efficient brute-force solution, and the point of the question is to find something more efficient than brute-force.

**Running Time.** Let $T(n)$ denote the running time of `mergesort(A, lo, hi)` where $n =$ `hi` $-$ `lo` $+ 1$. Then since we make 2 recursive calls of half the size and merge in linear time we have

$$T(n) = 2T(n/2) + O(n), \qquad T(1) = O(1)$$

And therefore the running time is $O(n \log n)$.

## Solving Recurrences

### Running Time Intuition: Trees

We can draw out trees of the recursive calls made by our algorithm and the work done at each stage to get intuition for the running time of our algorithm.

We write our runtime recurrences as

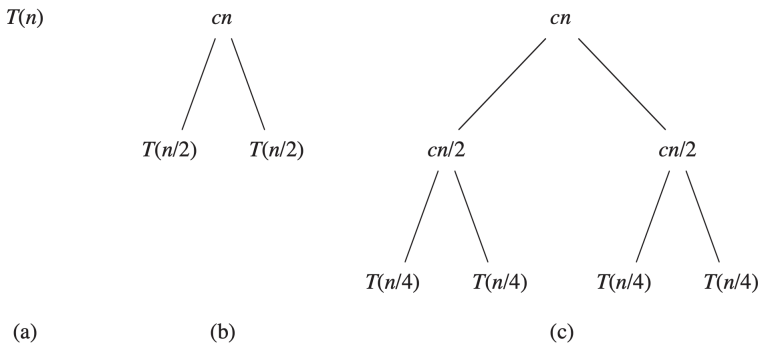
$$T(n) = a\,T(n/b) + f(n)$$

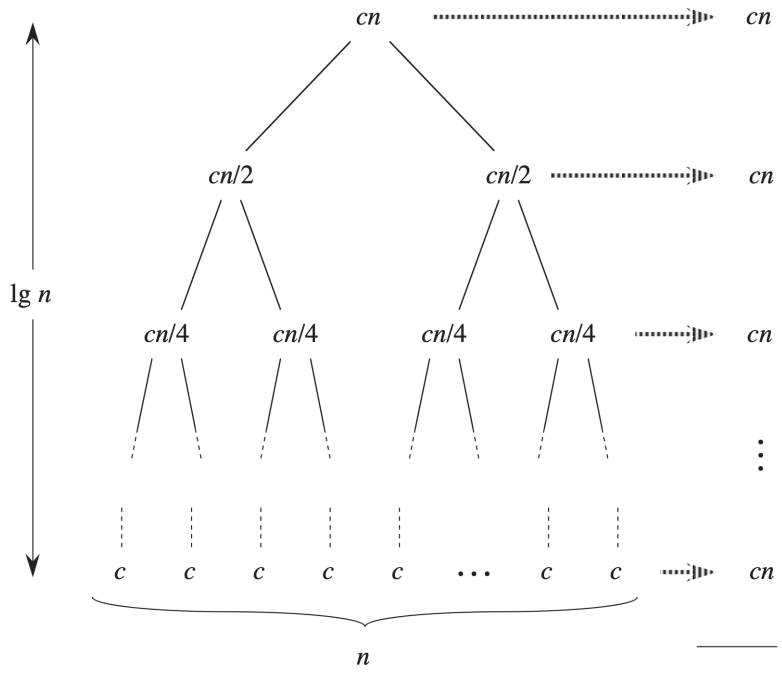
where $a$ is the number of subproblems that we make a call to, $n/b$ is the size of our subproblem, and $f(n)$ is the running time to divide and combine our subproblems.

MergeSort solves $a = 2$ subproblems. Each subproblem is of size $n/2$ (so $b = 2$). The running time to divide our solution is just the running time to compute `mid`, which is constant. The running time to combine requires looping through the first element of both subproblem solution, so is $\Theta(n) = f(n)$. Our base case is of size 1 and is constant runtime, $T(1) = O(1)$. Hence our recurrence for MergeSort is:

$$T(n) = 2T(n/2) + O(n), \qquad T(1) = O(1).$$

We construct recursion trees in order to understand the sum of the runtime from these recurrences. In constructing a recursion tree for the recurrence $T(n) = 2\,T(n/2) + O(n)$: Part (a) shows $T(n)$, which progressively expands in (b)–(d) to form the recursion tree. First we draw the two ($a$) recursive calls to subproblems of size $n/2$ that will run in time $T(n/2)$ (or $T(n/b)$). We then expand from $T(n/2)$ and its recursive calls, all the way down to the base cases. The fully expanded tree in part (d) has $\log_2(n) + 1$ levels (i.e., it has height $\log_2 n$, as indicated—this will depend on the parameter $b$), and we can see that each level contributes a total cost of $cn$. The total cost, therefore, is $cn \log_2 n + cn$, which is $\Theta(n \log_2 n)$.






### Proving Runtime Recurrences with Substitution

How do you prove this? Like induction! Your base case here would be $k = 1$, where $T(1) = 1$ by our algorithm's code. Our "inductive hypothesis" here will be for $k = n/2$, that the running time is $n/2 \log_2 n/2$. (It's very important to be careful about boundary conditions when handling base cases.)

Then substituting into our recurrence we get:

$$\begin{aligned} T(n) &\leq 2\left(c\lceil\frac{n}{2}\rceil \log_2 \lceil\frac{n}{2}\rceil\right) + n \\ &= cn \log_2 n - cn \log_2 2 + n \\ &= cn \log_2 n - cn + n \\ &\leq cn \log_2 n \\ &= O(n \log n). \end{aligned}$$

### The "Master Theorem" for Divide & Conquer Runtime

Recall that $a$ is the number of subproblems, $n/b$ is the size of our subproblem, and $f(n)$ is the running time to divide and combine our subproblems.

The master theorem tells us the running time for most recurrences as a function of the parameters $a$, $b$, and $f(n)$ mentioned above, although it does not actually cover every case, as there are polynomial gaps between the cases.

**Theorem 1** (The Master Theorem)**.** *Let $a \geq 1$ and $b > 1$ be constants, let $f(n)$ be a function, and let $T(n)$ be defined on the non-negative integers by the recurrence*

$$T(n) = a\,T(n/b) + f(n),$$

*where we interpret $n/b$ as $\lfloor n/b \rfloor$ or $\lceil n/b \rceil$. Then*

1. *If $f(n) = O(n^{\log_b a - \varepsilon})$ for a constant $\varepsilon > 0$, then*

$$T(n) = \Theta(n^{\log_b a}).$$

2. *If $f(n) = \Theta(n^{\log_b a})$, then*

$$T(n) = \Theta(n^{\log_b a} \log_2 n).$$

3. *If $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for a constant $\varepsilon > 0$ and $af(n/b) \leq cf(n)$ for a constant $c < 1$ and for all sufficiently large $n$, then*

$$T(n) = \Theta(f(n)).$$

Solve the following recurrences using the master theorem.

- $T(n) = 9T(n/3) + n$.

  Case 1. $a = 9, b = 3, f(n) = n$. $n^{\log_b a} = n^{\log_3 9} = \Theta(n^2)$. $f(n) = O(n^{\log_3 9 - \varepsilon}$ for $\varepsilon = 1$. Then $T(n) = \Theta(n^2)$.

- $T(n) = T(2n/3) + 1$.

  Case 2. $a = 1, b = 3/2, f(n) = 1$. $n^{\log_b a} = n^{\log_{3/2} 1} = n^0 = 1$. $f(n) = \Theta(n^{\log_b a} = \Theta(1)$. Then $T(n) = \Theta(\log n)$.

- $T(n) = 3T(n/4) + n \log n$.

  Case 3. $a = 3, b = 4, f(n) = n \log n$. $n^{\log_b a} = n^{\log_4 3} = \Theta(n^{0.793})$. $f(n) = \Omega(n^{\log_4 3 + \varepsilon}$ for $\varepsilon \approx 0.2$. For sufficiently large $n$, $af(n/b) = 3(n/4)\log(n/4) \leq cf(n)$ for $c = 3/4$). Then $T(n) = \Theta(n \log n)$.